

# CMB-lensing beyond the leading order: temperature and polarization anisotropies

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**Abstract.** We investigate the weak lensing corrections to the CMB temperature and polarization anisotropies. We consider all the effects beyond the leading order: post-Born corrections, LSS corrections and, for the polarization anisotropies, the correction due to the rotation of the polarization direction between the emission at the source and the detection at the observer. We show that the full next-to-leading order correction to the B-mode polarization is not negligible on small scales and is dominated by the contribution from the rotation, this is a new effect not taken in account in previous works. Considering vanishing primordial gravitational waves, the B-mode correction due to rotation is comparable to cosmic variance for  $\ell \gtrsim 3500$ , in contrast to all other spectra where the corrections are always below that threshold for a single multipole. Moreover, the sum of all the effects is larger than cosmic variance at high multipoles, showing that higher-order lensing corrections to B-mode polarization are in principle detectable.

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## 1 Introduction

The temperature and polarization anisotropies of the cosmic microwave background (CMB) are the most precious cosmological datasets. It is fair to say that virtually all high precision cosmological measurements involve the CMB. The reason for this is twofold: on the one hand there is excellent data available [1–8] and on the other hand CMB fluctuations are theoretically well understood and can be calculated perturbatively. The CMB success story is by no means over, we expect more precision data to arrive especially for polarization and reconstruction of the cosmic lens map [9, 10].

As it is well known, CMB fluctuations are lensed by foreground large scale structure (LSS) and this effect is rather large (up to 10% and more) on small scales [11–13]. Therefore the question is justified whether higher order contributions to lensing might be relevant. We naively expect them to be of the order of the square of the first order contribution, hence 1% and therefore it is necessary to include them as numerical CMB calculations [14–17] aim at a precision of 0.1%. On the other hand, present CMB codes do take into account some of the non-linearities by summing up a series of ‘ladder diagrams’ into an exponential [12, 13]. It is easy to check that including these non-linearities is requested to achieve the precision goal.

The question which we address in this paper is: what about the other non-linearities which are not included in this sum? Might they also be relevant? These are mainly contributions coming from the fact that the deflection angle of the photons at higher order can no longer be computed assuming the photons move along their unperturbed path, but the perturbation of the photon path has to be taken into account. These are the so-called ‘post-Born corrections’. We have already studied this problem for the temperature anisotropies in a previous paper [18]. The present paper is a follow up on that work. We complete the previous study by calculating also the effects on polarization. Furthermore, here we treat also the non-linearities of the matter distribution perturbatively. This is more consistent than just using a Halofit model [19, 20], as it allows us to correctly take into account the higher order statistics (3- and 4-point functions) assuming Gaussian first order perturbations.

In addition to our work, there have been three other publications on this topic [21–23]. In the first paper, an important cancellation which reduces the final result by more than an order of magnitude has been missed. In [22] our so called ‘third group’ terms, which vanish when assuming Gaussian statistics and are very relevant for the final result, are not included. In the most recent publication [23] these terms are included, but the rotation of the polarization which is induced by second order lensing is not considered. We discuss it here for the first time and we actually find that it is the dominant correction for  $B$ -polarization.

In this paper we present the methodology of our calculations and numerical results for the corrections of CMB temperature and polarization anisotropies by next-to-leading order lensing. In an accompanying letter [24] we discuss the relevance of our findings for future CMB experiments.

The paper is organized as follows. In the next section we summarize the small deflection angle approximation for CMB lensing beyond linear order, and present the expressions for the deflection angle up to third order. In Sect. 3 we translate the results into harmonic space, ‘ $\ell$  space’. We also compare the expressions for temperature anisotropies with the corresponding terms for the polarization spectra at all orders in perturbation theory. In Sect. 4 we briefly recollect the results for the post-Born corrections to the lensed power spectrum of the CMB temperature anisotropies first given in [18] considering also the non-Gaussian nature of the deflection angle at higher order. In Sect. 5 we evaluate the contributions from higher orders in the gravitational potential (or equivalently in the matter density) to corrections of the

lensed power spectrum of the CMB temperature and polarization anisotropies. Following [22, 23] we call them ‘LSS corrections’. In Sect. 6 we derive the last missing contribution coming from the fact that parallel transported polarization direction changes along the path of the photon from the source to the observer. This contribution which turns out to be very substantial has been missed in previous work. Our results are summarized in Sect. 7, where we evaluate the different contributions numerically considering a Halofit matter power spectrum. In Sect. 8 we conclude. Several technical aspects and calculations are presented in four appendices.

## 2 Weak lensing corrections beyond leading order in real space

We want to determine the effect of lensing on the CMB temperature and polarization anisotropies beyond the well studied leading order from first order perturbation theory [12, 13].

Following the derivation of the post-Born correction to temperature anisotropies in [18], we first generalize the results of [12, 13] writing the following relation between the lensed and unlensed temperature anisotropies  $\mathcal{M}$  and polarization tensor  $\mathcal{P}_{mn}$  of the photon field valid up to fourth order in the deflection angles  $\theta^{a(i)}$  (the superscript  $(i)$  denotes the order).

$$\begin{aligned}\tilde{\mathcal{M}}(x^a) \equiv \mathcal{M}(x^a + \delta\theta^a) &\simeq \mathcal{M}(x^a) + \sum_{i=1}^4 \theta^{b(i)} \nabla_b \mathcal{M}(x^a) + \frac{1}{2} \sum_{i+j \leq 4} \theta^{b(i)} \theta^{c(j)} \nabla_b \nabla_c \mathcal{M}(x^a) \\ &+ \frac{1}{6} \sum_{i+j+k \leq 4} \theta^{b(i)} \theta^{c(j)} \theta^{d(k)} \nabla_b \nabla_c \nabla_d \mathcal{M}(x^a) + \frac{1}{24} \theta^{b(1)} \theta^{c(1)} \theta^{d(1)} \theta^{e(1)} \nabla_b \nabla_c \nabla_d \nabla_e \mathcal{M}(x^a),\end{aligned}\tag{2.1}$$

$$\begin{aligned}\tilde{\mathcal{P}}_{mn}(x^a) \equiv \mathcal{P}_{mn}(x^a + \delta\theta^a) &\simeq \mathcal{P}_{mn}(x^a) + \sum_{i=1}^4 \theta^{b(i)} \nabla_b \mathcal{P}_{mn}(x^a) \\ &+ \frac{1}{2} \sum_{i+j \leq 4} \theta^{b(i)} \theta^{c(j)} \nabla_b \nabla_c \mathcal{P}_{mn}(x^a) + \frac{1}{6} \sum_{i+j+k \leq 4} \theta^{b(i)} \theta^{c(j)} \theta^{d(k)} \nabla_b \nabla_c \nabla_d \mathcal{P}_{mn}(x^a) \\ &+ \frac{1}{24} \theta^{b(1)} \theta^{c(1)} \theta^{d(1)} \theta^{e(1)} \nabla_b \nabla_c \nabla_d \nabla_e \mathcal{P}_{mn}(x^a).\end{aligned}\tag{2.2}$$

A consistent treatment of the polarization in the form of  $\mathcal{P}_{mn}$  or, using the Stokes parameters  $\mathcal{Q}$  and  $\mathcal{U}$ , in the form of  $\mathcal{P} = \mathcal{Q} + i\mathcal{U}$  and  $\bar{\mathcal{P}} = \mathcal{Q} - i\mathcal{U}$  has to consider that the polarization tensor is parallel-transported along the perturbed photon geodesics. Neglecting this effect (we shall add it at a second stage in Sect. 6) we can substitute  $\mathcal{P}_{mn}$  with  $\mathcal{P}$  and  $\bar{\mathcal{P}}$ . An over-bar denotes complex conjugation.

Following [18], we can then write

$$\tilde{\mathcal{M}}(x^a) \simeq \mathcal{A}^{(0)}(x^a) + \sum_{i=1}^4 \mathcal{A}^{(i)}(x^a) + \sum_{\substack{i+j \leq 4 \\ 1 \leq i \leq j}} \mathcal{A}^{(ij)}(x^a) + \sum_{\substack{i+j+k \leq 4 \\ 1 \leq i \leq j \leq k}} \mathcal{A}^{(ijk)}(x^a) + \mathcal{A}^{(1111)}(x^a),\tag{2.3}$$

$$\tilde{\mathcal{P}}(x^a) \simeq \mathcal{D}^{(0)}(x^a) + \sum_{i=1}^4 \mathcal{D}^{(i)}(x^a) + \sum_{\substack{i+j \leq 4 \\ 1 \leq i \leq j}} \mathcal{D}^{(ij)}(x^a) + \sum_{\substack{i+j+k \leq 4 \\ 1 \leq i \leq j \leq k}} \mathcal{D}^{(ijk)}(x^a) + \mathcal{D}^{(1111)}(x^a), \quad (2.4)$$

where

$$\mathcal{A}^{(i_1 i_2 \dots i_n)}(x^a) = \frac{\text{Perm}(i_1 i_2 \dots i_n)}{n!} \theta^{b(i_1)} \theta^{c(i_2)} \dots \nabla_b \nabla_c \dots \mathcal{M}(x^a), \quad (2.5)$$

$$\mathcal{D}^{(i_1 i_2 \dots i_n)}(x^a) = \frac{\text{Perm}(i_1 i_2 \dots i_n)}{n!} \theta^{b(i_1)} \theta^{c(i_2)} \dots \nabla_b \nabla_c \dots \mathcal{P}(x^a), \quad (2.6)$$

where  $\mathcal{A}^{(0)}(x^a) \equiv \mathcal{M}(x^a)$ ,  $\mathcal{D}^{(0)}(x^a) \equiv \mathcal{P}(x^a)$  and  $\text{Perm}(i_1 i_2 \dots i_n)$  denotes the number of permutation of the set  $(i_1 i_2 \dots i_n)$ .

We introduce also the Weyl potential

$$\Phi_W = \frac{1}{2} (\Phi + \Psi) \quad (2.7)$$

in terms of the Bardeen potentials  $\Phi$  and  $\Psi$ . The lensing potential  $\psi$  to the last scattering surface is then determined by

$$\psi(\mathbf{n}, z_s) = \frac{-2}{\eta_o - \eta_s} \int_{\eta_s}^{\eta_o} d\eta \frac{\eta - \eta_s}{\eta_o - \eta} \Phi_W((\eta - \eta_o)\mathbf{n}, \eta) = -2 \int_0^{r_s} dr' \frac{r_s - r'}{r_s r'} \Phi_W(-r'\mathbf{n}, \eta_o - r'), \quad (2.8)$$

where  $\mathbf{n}$  is the direction of photon propagation,  $\eta$  denotes conformal time and  $r$  the comoving distance,  $r = \eta_o - \eta$ , where  $\eta_o$  stands for present time. The index  $s$  indicates the corresponding quantity evaluated at the last scattering surface. The first order deflection angle is simply the gradient of the lensing potential [13, 25]. Beyond the linear order, we need to account also for the lensing of the direction  $\mathbf{n}$  on the path of the photon. Then one obtains the following expressions for the deflection angle up to third perturbative order [26]

$$\theta^{a(1)} = -2 \int_0^{r_s} dr' \frac{r_s - r'}{r_s r'} \nabla^a \Phi_W(r'), \quad (2.9)$$

$$\theta^{a(2)} = -2 \int_0^{r_s} dr' \frac{r_s - r'}{r_s r'} \left[ \nabla^a \Phi_W^{(2)}(r') + \nabla_b \nabla^a \Phi_W(r') \theta^{b(1)}(r') \right], \quad (2.10)$$

$$\begin{aligned} \theta^{a(3)} = & -2 \int_0^{r_s} dr' \frac{r_s - r'}{r_s r'} \left[ \nabla^a \Phi_W^{(3)}(r') + \nabla_b \nabla^a \Phi_W(r') \theta^{b(2)}(r') + \nabla_b \nabla^a \Phi_W^{(2)}(r') \theta^{b(1)}(r') \right. \\ & \left. + \frac{1}{2} \nabla_b \nabla_c \nabla^a \Phi_W(r') \theta^{b(1)}(r') \theta^{c(1)}(r') \right]. \end{aligned} \quad (2.11)$$

Latin letters  $a, b, c, d$  run over the two directions on the sphere. In Eqs. (2.9-2.11) we consider the terms with the maximal number of transverse derivatives, including the ones that come from expanding the Weyl potential,  $\Phi_W$ , to higher order.

Let us also remind that the Taylor expansion in Eqs. (2.1) and (2.2) holds in the approximation of small deflection angles, i.e. when the deflection angle is much smaller than the angular separations related to a given  $C_\ell$ . This is valid for an angular separation of about 4.5 arc minutes which corresponds to  $\ell \lesssim 2500$  (see [11–13]). In this work, we adopt the small deflection angle approximation for the second and third order deflection angles only, which

are much smaller than this value, as a consequence our results are valid to much higher  $\ell$ s and we can safely present them up to  $\ell = 3500$ .

### 3 Weak lensing corrections of the power spectra

We evaluate the lensing correction to the angular power spectra  $C_\ell^{\mathcal{M}}$ ,  $C_\ell^{\mathcal{EM}}$ ,  $C_\ell^{\mathcal{E}}$  and  $C_\ell^{\mathcal{B}}$  in the flat sky limit. In this approximation, see e.g. [13], we replace the combination  $(\ell, m)$  with a 2-dimensional vector  $\ell$ . Therefore, the angular position is then the 2-dimensional Fourier transform of the position in  $\ell$  space at redshift  $z$ . For a generic variable  $Y(z, \mathbf{x})$  we have

$$Y(z, \mathbf{x}) = \frac{1}{2\pi} \int d^2\ell Y(z, \ell) e^{-i\ell \cdot \mathbf{x}}, \quad (3.1)$$

and

$$\langle Y(z_1, \ell) \bar{Y}(z_2, \ell') \rangle = \delta(\ell - \ell') C_\ell^Y(z_1, z_2), \quad (3.2)$$

while for polarization we have

$$\mathcal{P}(z, \mathbf{x}) = -\frac{1}{2\pi} \int d^2\ell [\mathcal{E}(z, \ell) + i\mathcal{B}(z, \ell)] e^{-2i\varphi_\ell} e^{-i\ell \cdot \mathbf{x}}, \quad (3.3)$$

with

$$\begin{aligned} \langle \mathcal{E}(z_s, \ell) \bar{\mathcal{M}}(z_s, \ell') \rangle &= \delta(\ell - \ell') C_\ell^{\mathcal{EM}}(z_s), \\ \langle \mathcal{E}(z_s, \ell) \bar{\mathcal{E}}(z_s, \ell') \rangle &= \delta(\ell - \ell') C_\ell^{\mathcal{E}}(z_s), \\ \langle \mathcal{B}(z_s, \ell) \bar{\mathcal{B}}(z_s, \ell') \rangle &= \delta(\ell - \ell') C_\ell^{\mathcal{B}}(z_s), \\ \langle \mathcal{B}(z_s, \ell) \bar{\mathcal{M}}(z_s, \ell') \rangle &= 0, \\ \langle \mathcal{B}(z_s, \ell) \bar{\mathcal{E}}(z_s, \ell') \rangle &= 0. \end{aligned} \quad (3.4)$$

We follow the notation of [27, 28] to determine the angular power spectra defined above and we introduce the (3-dimensional) initial curvature power spectrum

$$\langle R_{\text{in}}(\mathbf{k}) \bar{R}_{\text{in}}(\mathbf{k}') \rangle = \delta_D(\mathbf{k} - \mathbf{k}') P_R(k). \quad (3.5)$$

(In both 2- and 3-dimensional Fourier transforms we adopt the unitary Fourier transform normalization, so there are no factors of  $2\pi$  in this formula as well as in Eqs. (3.2) and (3.4).)

For a given linear perturbation variable  $A$  we define its transfer function  $T_A(z, k)$  normalized to the initial curvature perturbation by

$$A(z, \mathbf{k}) = T_A(z, k) R_{\text{in}}(\mathbf{k}), \quad (3.6)$$

and an angular power spectrum will be then determined by

$$C_\ell^{AB}(z_1, z_2) = 4\pi \int \frac{dk}{k} \mathcal{P}_R(k) \Delta_\ell^A(z_1, k) \Delta_\ell^B(z_2, k) = \frac{2}{\pi} \int dk k^2 P_R(k) \Delta_\ell^A(z_1, k) \Delta_\ell^B(z_2, k), \quad (3.7)$$

where  $\mathcal{P}_R(k) = \frac{k^3}{2\pi^2} P_R(k)$  is the dimensionless primordial power spectrum, and  $\Delta_\ell^A(z, k)$  denotes the transfer function in angular and redshift space for the variable  $A$ . For instance, by considering  $A = B = \Phi_W$  and  $A = B = \psi$  we obtain that (setting  $C_\ell^{\Psi_W}(z, z') \equiv C_\ell^W(z, z')$ )

$$C_\ell^W(z, z') = \frac{1}{2\pi} \int dk k^2 P_R(k) [T_{\Psi+\Phi}(k, z) j_\ell(kr)] [T_{\Psi+\Phi}(k, z') j_\ell(kr')] , \quad (3.8)$$

$$C_\ell^\psi(z, z') = \frac{2}{\pi} \int dk k^2 P_R(k) \left[ \int_0^r dr_1 \frac{r - r_1}{rr_1} T_{\Psi+\Phi}(k, z_1) j_\ell(kr_1) \right] \\ \times \left[ \int_0^{r'} dr_2 \frac{r' - r_2}{r'r_2} T_{\Psi+\Phi}(k, z_2) j_\ell(kr_2) \right] , \quad (3.9)$$

where  $j_\ell$  denotes a spherical Bessel function of order  $\ell$ . As before,  $r \equiv \eta_o - \eta$  is the comoving distance to redshift  $z$ , and analogously  $r', r_1, r_2$  denote the distances to redshifts  $z', z_1, z_2$ . Above and hereafter, we define  $z = z(r)$ ,  $z' = z(r')$ , etc..

Hereafter, in order to numerically evaluate the next-to-leading order lensing contributions to the CMB temperature and polarization anisotropies, we will apply the Limber approximation [29–31]. We remark that this approximation works very well for CMB lensing. Indeed, CMB lensing is appreciable only for  $\ell > 100$ , where the Limber approximation is very close to the exact solution.

Following [32], the Limber approximation can be written as

$$\frac{2}{\pi} \int dk k^2 f(k) j_\ell(kx_1) j_\ell(kx_2) \simeq \frac{\delta_D(x_1 - x_2)}{x_1^2} f\left(\frac{\ell + 1/2}{x_1}\right) , \quad (3.10)$$

where  $f(k)$  should be a smooth, not strongly oscillating function of  $k$  which decreases sufficiently rapidly for  $k \rightarrow \infty$  (more precisely,  $f(k)$  has to decrease faster than  $1/k$  for  $k > \ell/x$ ). Using this approximation, one can then obtain the Limber-approximated  $C_\ell^W$  and  $C_\ell^\psi$  (see [18] for details).

Starting with the definitions (3.1) and (3.3), we can transform Eqs. (2.3) and (2.4) into  $\ell$  space where they become (see [18] for details)

$$\tilde{\mathcal{M}}(z_s, \ell) \simeq \mathcal{A}^{(0)}(\ell) + \sum_{i=1}^4 \mathcal{A}^{(i)}(\ell) + \sum_{\substack{i+j \leq 4 \\ 1 \leq i \leq j}} \mathcal{A}^{(ij)}(\ell) + \sum_{\substack{i+j+k \leq 4 \\ 1 \leq i \leq j \leq k}} \mathcal{A}^{(ijk)}(\ell) + \mathcal{A}^{(1111)}(\ell) , \quad (3.11)$$

$$\tilde{\mathcal{P}}(z_s, \ell) \simeq \mathcal{D}^{(0)}(\ell) + \sum_{i=1}^4 \mathcal{D}^{(i)}(\ell) + \sum_{\substack{i+j \leq 4 \\ 1 \leq i \leq j}} \mathcal{D}^{(ij)}(\ell) + \sum_{\substack{i+j+k \leq 4 \\ 1 \leq i \leq j \leq k}} \mathcal{D}^{(ijk)}(\ell) + \mathcal{D}^{(1111)}(\ell) , \quad (3.12)$$

where we drop the redshift dependence for simplicity on the right hand side, and we have

$$\mathcal{D}^{(0)}(z_s, \ell) \equiv \mathcal{P}(z_s, \ell) = \frac{1}{2\pi} \int d^2x \mathcal{P}(z, \mathbf{x}) e^{i\ell \cdot \mathbf{x}} = -[\mathcal{E}(z, \ell) + i\mathcal{B}(z, \ell)] e^{-2i\varphi_\ell} . \quad (3.13)$$

To evaluate the lensing corrections at next-to-leading order we have now to calculate the  $\ell$  space expressions for the terms  $\mathcal{A}^{(i, \dots)}$  and  $\mathcal{D}^{(i, \dots)}$ . The expressions for  $\mathcal{A}^{(i, \dots)}$  considering

at next-to-leading order only the post-Born corrections were determined in [18]. Starting from these results (see Appendix A of [18]), and from the results of Sect. 5 for the LSS corrections, one can easily find the corresponding expressions for  $\mathcal{D}^{(i,\dots)}$  both at leading and next-to-leading order. They are obtained from the  $\mathcal{A}^{(i,\dots)}$  by the substitution

$$\mathcal{M}(z_s, \ell) \rightarrow -[\mathcal{E}(z_s, \ell) + i\mathcal{B}(z_s, \ell)] e^{-2i\varphi_\ell}, \quad (3.14)$$

performed for any  $\mathcal{M}(z_s, \ell)$  inside the integrals. For completeness, we report them in Appendix A. This is very useful as it means, comparing Eq. (3.12) with Eq. (3.11) and using Eq. (3.4), that the lensing corrections at the next-to-leading order of  $C_\ell^{\mathcal{E}\mathcal{M}}$ ,  $C_\ell^{\mathcal{E}}$  and  $C_\ell^{\mathcal{B}}$  can be obtained, as the leading lensing corrections (see [12, 13]), by using the results for  $C_\ell^{\mathcal{M}}$  by a series of simple substitutions (see also [23]). Namely, we find that the corrections to  $C_\ell^{\mathcal{E}\mathcal{M}}$  are obtained by substituting

$$C_\ell^{\mathcal{M}}(z_s) \rightarrow C_\ell^{\mathcal{E}\mathcal{M}}(z_s) \quad , \quad \hat{C}_{\ell_1}^{\mathcal{M}}(z_s) \rightarrow C_{\ell_1}^{\mathcal{E}\mathcal{M}}(z_s) \cos[2(\varphi_{\ell_1} - \varphi_\ell)], \quad (3.15)$$

the corrections to  $C_\ell^{\mathcal{E}}$  by substituting

$$C_\ell^{\mathcal{M}}(z_s) \rightarrow C_\ell^{\mathcal{E}}(z_s) \quad , \quad \hat{C}_{\ell_1}^{\mathcal{M}}(z_s) \rightarrow C_{\ell_1}^{\mathcal{E}}(z_s) \cos^2[2(\varphi_{\ell_1} - \varphi_\ell)] + C_{\ell_1}^{\mathcal{B}}(z_s) \sin^2[2(\varphi_{\ell_1} - \varphi_\ell)], \quad (3.16)$$

and, finally, the corrections to  $C_\ell^{\mathcal{B}}$  by substituting

$$C_\ell^{\mathcal{M}}(z_s) \rightarrow C_\ell^{\mathcal{B}}(z_s) \quad , \quad \hat{C}_{\ell_1}^{\mathcal{M}}(z_s) \rightarrow C_{\ell_1}^{\mathcal{E}}(z_s) \sin^2[2(\varphi_{\ell_1} - \varphi_\ell)] + C_{\ell_1}^{\mathcal{B}}(z_s) \cos^2[2(\varphi_{\ell_1} - \varphi_\ell)], \quad (3.17)$$

where we use a  $\hat{\phantom{x}}$  to indicate the  $C_\ell^{\mathcal{M}}$  that are inside an integral (for completeness, we present more details in Appendix B).

At this point, let us briefly recall our approach to obtain the lensing correction to the temperature anisotropies beyond leading order (see [18] for details). Following [18], we have that

$$\langle \tilde{\mathcal{M}}(\ell) \bar{\tilde{\mathcal{M}}}(\ell') \rangle = \langle \mathcal{A}(\ell) \bar{\mathcal{A}}(\ell') \rangle, \quad (3.18)$$

where

$$\mathcal{A}(\ell) = \mathcal{A}^{(0)}(\ell) + \sum_{i=1}^4 \mathcal{A}^{(i)}(\ell) + \sum_{\substack{i+j \leq 4 \\ 1 \leq i \leq j}} \mathcal{A}^{(ij)}(\ell) + \sum_{\substack{i+j+k \leq 4 \\ 1 \leq i \leq j \leq k}} \mathcal{A}^{(ijk)}(\ell) + \mathcal{A}^{(1111)}(\ell). \quad (3.19)$$

We now introduce  $C_\ell^{(i,\dots,j,\dots)}$  defined by

$$\begin{aligned} \delta(\ell - \ell') C_\ell^{(ij,\dots,ij,\dots)} &= \langle \mathcal{A}^{(ij,\dots)}(\ell) \bar{\mathcal{A}}^{(ij,\dots)}(\ell') \rangle, \\ \delta(\ell - \ell') C_\ell^{(ij,\dots,i'j',\dots)} &= \langle \mathcal{A}^{(ij,\dots)}(\ell) \bar{\mathcal{A}}^{(i'j',\dots)}(\ell') \rangle + \langle \mathcal{A}^{(i'j',\dots)}(\ell) \bar{\mathcal{A}}^{(ij,\dots)}(\ell') \rangle, \end{aligned} \quad (3.20)$$

where the last definition applies when the coefficients  $(ij \dots)$  and  $(i'j' \dots)$  are not identical. The delta Dirac function  $\delta(\ell - \ell')$  is a consequence of statistical isotropy. By omitting terms



of higher than fourth order in the Weyl potential and terms that vanish as a consequence of Wick's theorem (odd number of Weyl potentials), we obtain

$$\begin{aligned}\tilde{C}_\ell^{\mathcal{M}} = & C_\ell^{\mathcal{M}} + C_\ell^{(0,2)} + C_\ell^{(0,11)} + C_\ell^{(1,1)} + C_\ell^{(0,4)} + C_\ell^{(0,13)} + C_\ell^{(0,22)} + C_\ell^{(0,112)} + C_\ell^{(0,1111)} \\ & + C_\ell^{(1,3)} + C_\ell^{(2,2)} + C_\ell^{(1,12)} + C_\ell^{(1,111)} + C_\ell^{(2,11)} + C_\ell^{(11,11)},\end{aligned}\quad (3.21)$$

where  $C_\ell^{(0,0)} \equiv C_\ell^{\mathcal{M}}$  is the unlensed power spectrum. The terms  $C_\ell^{(0,2)}$ ,  $C_\ell^{(0,4)}$  and  $C_\ell^{(0,112)}$ , containing an odd number of deflection angles from only one direction, are identically zero as a consequence of statistical isotropy. This was shown explicitly for the post-Born part of  $C_\ell^{(0,112)}$  in [18] and for the second order contribution  $C_\ell^{(0,2)}$  in [33].

Furthermore, making use of the Gaussian statistics of the first order deflection angle, the full correction from first order deflection angles alone, to the unlensed  $C_\ell^{\mathcal{M}}$ , i.e. all the terms above containing only 0's and 1's, can be fully re-summed [11–13]. Denoting this sum by  $\tilde{C}_\ell^{\mathcal{M}(1)}$  we have

$$\tilde{C}_\ell^{\mathcal{M}(1)} = \int dr r J_0(\ell r) \int \frac{d^2 \ell'}{(2\pi)^2} C_{\ell'}^{\mathcal{M}} e^{-i\ell' \cdot \mathbf{r}} \exp \left[ -\frac{\ell'^2}{2} (A_0(0) - A_0(r) + A_2(r) \cos(2\phi)) \right], \quad (3.22)$$

with

$$A_0(r) = \int \frac{d\ell}{2\pi} \ell^3 C_\ell^\psi J_0(r\ell) \quad , \quad A_2(r) = \int \frac{d\ell}{2\pi} \ell^3 C_\ell^\psi J_2(r\ell) . \quad (3.23)$$

and where  $J_0$  and  $J_2$  are the Bessel functions of order zero and two.

We now write

$$\tilde{C}_\ell^{\mathcal{M}} = \tilde{C}_\ell^{\mathcal{M}(1)} + \Delta C_\ell^{(2)} + \Delta C_\ell^{(3)} \quad (3.24)$$

where (neglecting vanishing contributions)

$$\Delta C_\ell^{(2)} = C_\ell^{(0,13)} + C_\ell^{(0,22)} + C_\ell^{(1,3)} + C_\ell^{(2,2)}, \quad (3.25)$$

$$\Delta C_\ell^{(3)} = C_\ell^{(1,12)} + C_\ell^{(2,11)}. \quad (3.26)$$

As already mentioned,  $\tilde{C}_\ell^{\mathcal{M}(1)}$  denotes the well known resummed correction from the first order deflection angle [11–13], which is computed in standard CMB-codes [14, 15].  $\Delta C_\ell^{(2)}$  and  $\Delta C_\ell^{(3)}$  denote corrections involving two or three deflection angles respectively, at least one of them beyond the Born approximation or with and higher order Weyl potential. With a slight abuse of language we call them the Gaussian and non-Gaussian contribution of the deflection angle or, as in [18], the second and third group respectively. Even though the contributions to the second group are not Gaussian, they would be present also if the higher order deflection angles would be Gaussian. Terms of the third group, however, would vanish for Gaussian higher order deflection angles. Note that even though the number of deflection angles is odd in the third group, statistical isotropy does not require it to vanish as (in the correlation function picture) there is in addition the angle between the two directions  $\mathbf{n}_1$  and  $\mathbf{n}_2$  which can be employed to 'pair up' all the angles. If the deflections are all attached to one of these

two directions this additional angle is no longer present and a term of the form  $C_\ell^{(0,n_1\cdots n_{2j+1})}$  has to vanish due to statistical isotropy, while a term of the form  $C_\ell^{(n_1\cdots n_k, n_{k+1}\cdots n_{2j+1})}$  with  $k > 0$  does not. Here we of course always assume that CMB anisotropies and deflection angles are uncorrelated as the latter come from much lower redshifts.

Furthermore, within the Limber approximation which is very accurate for these small corrections relevant only at high  $\ell$  the two contributions  $C_\ell^{(0,13)}$  and  $C_\ell^{(0,22)}$  coming from the post-Born part of the deflection angle exactly cancel,  $C_\ell^{(0,13)} = -C_\ell^{(0,22)}$ . This is no longer so when we consider the LSS contributions to these terms, see Sect. 5 below.

## 4 Post-Born contributions

Let us first recall the results for the post-Born lensing corrections obtained in [18] for the temperature anisotropies. The results for polarization spectra can then be obtained as illustrated in the previous section.

### 4.1 Second group

The second group, where we study the leading post-Born corrections coming from the deflection angles up to third order when these appear in pairs like  $\langle \theta^{a(2)} \theta^{b(2)} \rangle$  and  $\langle \theta^{a(1)} \theta^{b(3)} \rangle$ , is given by

$$\begin{aligned} C_{\ell,pB}^{(1,3)} = & - \int \frac{d^2 \ell_1}{(2\pi)^2} \int \frac{d^2 \ell_2}{(2\pi)^2} [(\ell - \ell_1) \cdot \ell_1]^2 [(\ell - \ell_1) \cdot \ell_2]^2 \hat{C}_{\ell_1}^{\mathcal{M}}(z_s) \\ & \times \int_0^{r_s} dr' \frac{(r_s - r')^2}{r_s^2 r'^4} C_{\ell_2}^\psi(z', z') \\ & \times P_R \left( \frac{|\ell - \ell_1| + 1/2}{r'} \right) \left[ T_{\Psi+\Phi} \left( \frac{|\ell - \ell_1| + 1/2}{r'}, z' \right) \right]^2, \end{aligned} \quad (4.1)$$

$$\begin{aligned} C_{\ell,pB}^{(2,2)} = & \int \frac{d^2 \ell_1}{(2\pi)^2} \int \frac{d^2 \ell_2}{(2\pi)^2} [(\ell - \ell_1 + \ell_2) \cdot \ell_1]^2 [(\ell - \ell_1 + \ell_2) \cdot \ell_2]^2 \hat{C}_{\ell_1}^{\mathcal{M}}(z_s) \\ & \times \int_0^{r_s} dr' \frac{(r_s - r')^2}{r_s^2 r'^4} C_{\ell_2}^\psi(z', z') \\ & \times P_R \left( \frac{|\ell - \ell_1 + \ell_2| + 1/2}{r'} \right) \left[ T_{\Psi+\Phi} \left( \frac{|\ell - \ell_1 + \ell_2| + 1/2}{r'}, z' \right) \right]^2. \end{aligned} \quad (4.2)$$

### 4.2 Third group

The third group, where we consider terms with three deflection angles which do not vanish due to the non-Gaussian statistic of  $\theta^{a(2)}$ , is given by

$$\begin{aligned} C_{\ell,pB}^{(1,12)} = & -2 \int \frac{d^2 \ell_1}{(2\pi)^2} \int \frac{d^2 \ell_2}{(2\pi)^2} (\ell_1 \cdot \ell_2) [(\ell - \ell_1) \cdot \ell_2] [(\ell - \ell_1) \cdot \ell_1]^2 \\ & \times \hat{C}_{\ell_1}^{\mathcal{M}}(z_s) \int_0^{r_s} dr' \frac{(r_s - r')^2}{r_s^2 r'^4} P_R \left( \frac{|\ell - \ell_1| + 1/2}{r'} \right) \\ & \times \left[ T_{\Psi+\Phi} \left( \frac{|\ell - \ell_1| + 1/2}{r'}, z' \right) \right]^2 C_{\ell_2}^\psi(z_s, z'), \end{aligned} \quad (4.3)$$

$$\begin{aligned}
C_{\ell,pB}^{(2,11)} &= 2 \int \frac{d^2\ell_1}{(2\pi)^2} \int \frac{d^2\ell_2}{(2\pi)^2} (\ell_1 \cdot \ell_2) [(\ell - \ell_1 + \ell_2) \cdot \ell_2] [(\ell - \ell_1 + \ell_2) \cdot \ell_1]^2 \\
&\times \hat{C}_{\ell_1}^{\mathcal{M}}(z_s) \int_0^{r_s} dr' \frac{(r_s - r')^2}{r_s^2 r'^4} P_R \left( \frac{|\ell - \ell_1 + \ell_2| + 1/2}{r'} \right) \\
&\times \left[ T_{\Psi+\Phi} \left( \frac{|\ell - \ell_1 + \ell_2| + 1/2}{r'}, z' \right) \right]^2 C_{\ell_2}^{\psi}(z_s, z') .
\end{aligned} \tag{4.4}$$

Like for the temperature anisotropies (see [18]), also for the polarization spectra, the contributions above, within each group, partially erase each other. In the range of integration where  $|\ell - \ell_1 + \ell_2| \simeq |\ell - \ell_1|$  the integrands in Eqs. (4.1) and (4.2) (as well as the ones in Eqs. (4.3) and (4.4)) are nearly identical and the corresponding contributions partially cancel (see [18] for details and a physical interpretation).

## 5 LSS contributions

In this section we determine the next-to-leading order corrections to CMB lensing coming from higher order corrections of the Weyl potential (the so-called LSS contributions, see also [23]).

We want to determine the LSS contributions to the deflection angle up to third order. As one sees from Eqs. (2.10) and (2.11), this requires  $\Phi_W^{(2)}$  and  $\Phi_W^{(3)}$ . We use the Newtonian approximations to  $\Phi_W$  which are very accurate on largely sub-horizon scales,  $k/\mathcal{H} \gg 1$ , and in a matter dominated regime. They are given by (see for example [34])

$$\Phi_W^{(2)}(\mathbf{k}, \eta) = -\frac{3\mathcal{H}^2\Omega_m(\eta)}{2k^2}\delta^{(2)}(\mathbf{k}, \eta), \tag{5.1}$$

$$\delta^{(2)}(\mathbf{k}, \eta) = \frac{1}{(2\pi)^{3/2}} \int d^3k_1 d^3k_2 \delta_D(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) F_2(\mathbf{k}_1, \mathbf{k}_2) \delta(\mathbf{k}_1, \eta) \delta(\mathbf{k}_2, \eta), \tag{5.2}$$

$$F_2(\mathbf{k}_1, \mathbf{k}_2) = \frac{5}{7} + \frac{1}{2} \frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{k_1 k_2} \left( \frac{k_1}{k_2} + \frac{k_2}{k_1} \right) + \frac{2}{7} \left( \frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{k_1 k_2} \right)^2, \tag{5.3}$$

and [35, 36]

$$\Phi_W^{(3)}(\mathbf{k}, \eta) = -\frac{3\mathcal{H}^2\Omega_m(\eta)}{2k^2}\delta^{(3)}(\mathbf{k}, \eta), \tag{5.4}$$

$$\begin{aligned}
\delta^{(3)}(\mathbf{k}, \eta) &= \frac{1}{(2\pi)^3} \int d^3k_1 d^3k_2 d^3k_3 \delta_D(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) F_3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \\
&\times \delta(\mathbf{k}_1, \eta) \delta(\mathbf{k}_2, \eta) \delta(\mathbf{k}_3, \eta),
\end{aligned} \tag{5.5}$$

$$\begin{aligned}
F_3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) &= \frac{1}{18} \{ G_2(\mathbf{k}_1, \mathbf{k}_2) [7\alpha(\mathbf{k}_1 + \mathbf{k}_2, \mathbf{k}_3) + 4\beta(\mathbf{k}_1 + \mathbf{k}_2, \mathbf{k}_3)] \\
&+ 7\alpha(\mathbf{k}_1, \mathbf{k}_2 + \mathbf{k}_3) F_2(\mathbf{k}_2, \mathbf{k}_3) \},
\end{aligned} \tag{5.6}$$

with

$$\alpha(\mathbf{k}, \mathbf{k}') = \frac{(\mathbf{k} + \mathbf{k}') \cdot \mathbf{k}}{k^2}, \quad \beta(\mathbf{k}, \mathbf{k}') = \frac{(\mathbf{k} + \mathbf{k}')^2 \mathbf{k} \cdot \mathbf{k}'}{2k^2 k'^2}. \tag{5.7}$$

We now write explicit formulas for the case of temperature anisotropies, the corresponding expressions for E- and B-modes are obtained from the temperature results using the substitutions in Eqs. (3.15)-(3.17).

### 5.1 Second group

Let us first evaluate the impact of the LSS corrections on our second group. As we will show explicitly in the follow, within the Limber approximation the LSS contribution to the second group is already included when we consider an Halofit model in evaluating the leading first order contribution. Namely, it is equivalent to take the leading lensing correction, obtained from first order deflection angle, and consider in the  $C_\ell^\psi$  the higher order contributions to the gravitational potential (i.e., considering an higher order power spectrum).

To show this we write the deflection angles up to third order in terms of the 2-dimensional Fourier transform of the Weyl potential including also the LSS contributions from  $\Phi_W^{(2)}$  and  $\Phi_W^{(3)}$ . In general, an angle  $\theta^{a(n)}$  contains a part which depends only on the first order Weyl potential and a second part which depends on higher order corrections to the Weyl potential, up to third order these are  $\Phi_W^{(2)}$  and  $\Phi_W^{(3)}$ . The first part is the one evaluated in [18], let us call it  $\theta_{St}^{a(n)}$ , while we call the second part  $\theta_{LSS}^{a(n)}$ . Up to third order, the second part is given by

$$\theta_{LSS}^{a(2)}(\mathbf{x}) = \frac{i}{\pi} \int d^2\ell \int_0^{r_s} dr \frac{r_s - r}{r_s r} \ell^a \Phi_W^{(2)}(r, \ell) e^{-i\ell \cdot \mathbf{x}}, \quad (5.8)$$

$$\begin{aligned} \theta_{LSS}^{a(3)}(\mathbf{x}) = & \frac{i}{\pi} \int d^2\ell \int_0^{r_s} dr \frac{r_s - r}{r_s r} \ell^a \Phi_W^{(3)}(r, \ell) e^{-i\ell \cdot \mathbf{x}} \\ & + \frac{i}{\pi^2} \int d^2\ell_1 \int d^2\ell_2 \int_0^{r_s} dr \frac{r_s - r}{r_s r} \left( \ell_1^a \ell_{1b} \Phi_W^{(2)}(r, \ell_1) e^{-i\ell_1 \cdot \mathbf{x}} \right) \\ & \times \int_0^r dr' \frac{r - r'}{r r'} \ell_2^b \Phi_W(r', \ell_2) e^{-i\ell_2 \cdot \mathbf{x}} \\ & + \frac{i}{\pi^2} \int d^2\ell_1 \int d^2\ell_2 \int_0^{r_s} dr \frac{r_s - r}{r_s r} \left( \ell_1^a \ell_{1b} \Phi_W(r, \ell_1) e^{-i\ell_1 \cdot \mathbf{x}} \right) \\ & \times \int_0^r dr' \frac{r - r'}{r r'} \ell_2^b \Phi_W^{(2)}(r', \ell_2) e^{-i\ell_2 \cdot \mathbf{x}}. \end{aligned} \quad (5.9)$$

The LSS corrections to the second group contribute to  $C_\ell^{(0,22)}$ ,  $C_\ell^{(0,13)}$ ,  $C_\ell^{(2,2)}$  and  $C_\ell^{(1,3)}$ . To evaluate them we calculate the contribution of  $\Phi_W^{(2)}$  and  $\Phi_W^{(3)}$  to  $\mathcal{A}^{(2)}(\ell)$ ,  $\mathcal{A}^{(3)}(\ell)$ ,  $\mathcal{A}^{(13)}(\ell)$  and  $\mathcal{A}^{(22)}(\ell)$ . Following [18], we obtain

$$\begin{aligned} \mathcal{A}_{LSS}^{(2)}(\ell) &= \frac{1}{2\pi} \int d^2x \theta_{LSS}^{a(2)} \nabla_a \mathcal{M} e^{i\ell \cdot \mathbf{x}} \\ &= \frac{1}{\pi} \int d^2\ell_2 [(\ell - \ell_2) \cdot \ell_2] \int_0^{r_s} dr \frac{r_s - r}{r_s r} \Phi_W^{(2)}(r, \ell - \ell_2) \mathcal{M}(r_s, \ell_2), \end{aligned} \quad (5.10)$$

$$\begin{aligned}
\mathcal{A}_{\text{LSS}}^{(3)}(\ell) &= \frac{1}{2\pi} \int d^2x \theta_{\text{LSS}}^{a(3)} \nabla_a \mathcal{M} e^{i\ell \cdot \mathbf{x}} \\
&= \frac{1}{\pi} \int d^2\ell_2 [(\ell - \ell_2) \cdot \ell_2] \int_0^{r_s} dr \frac{r_s - r}{r_s r} \Phi_W^{(3)}(r, \ell - \ell_2) \mathcal{M}(r_s, \ell_2) \\
&\quad - \frac{1}{\pi^2} \int d^2\ell_2 \int d^2\ell_3 [(\ell + \ell_2 - \ell_3) \cdot \ell_3] [(\ell + \ell_2 - \ell_3) \cdot \ell_2] \int_0^{r_s} dr \frac{r_s - r}{r_s r} \\
&\quad \times \int_0^r dr' \frac{r - r'}{r r'} \left[ \Phi_W(r, \ell + \ell_2 - \ell_3) \bar{\Phi}_W^{(2)}(r', \ell_2) \right. \\
&\quad \left. + \Phi_W^{(2)}(r, \ell + \ell_2 - \ell_3) \bar{\Phi}_W(r', \ell_2) \right] \mathcal{M}(r_s, \ell_3), \tag{5.11}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_{\text{LSS}}^{(13)}(\ell) &= \frac{1}{2\pi} \int d^2x \theta_{\text{LSS}}^{a(1)} \theta_{\text{LSS}}^{b(3)} \nabla_a \nabla_b \mathcal{M} e^{i\ell \cdot \mathbf{x}} \\
&= -\frac{1}{\pi^2} \int d^2\ell_2 \int d^2\ell_3 [(\ell + \ell_2 - \ell_3) \cdot \ell_3] (\ell_2 \cdot \ell_3) \\
&\quad \times \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^{r_s} dr' \frac{r_s - r'}{r_s r'} \Phi_W(r, \ell + \ell_2 - \ell_3) \bar{\Phi}_W^{(3)}(r', \ell_3) \mathcal{M}(r_s, \ell_3) \\
&\quad + \frac{1}{\pi^3} \int d^2\ell_2 \int d^2\ell_3 \int d^2\ell_4 [(\ell - \ell_2 - \ell_3 - \ell_4) \cdot \ell_4] (\ell_4 \cdot \ell_2) (\ell_3 \cdot \ell_2) \\
&\quad \times \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^{r_s} dr' \frac{r_s - r'}{r_s r'} \int_0^{r'} dr'' \frac{r' - r''}{r' r''} \\
&\quad \times \Phi_W(r, \ell - \ell_2 - \ell_3 - \ell_4) \left[ \Phi_W(r', \ell_2) \Phi_W^{(2)}(r'', \ell_3) \right. \\
&\quad \left. + \Phi_W^{(2)}(r', \ell_2) \Phi_W(r'', \ell_3) \right] \mathcal{M}(r_s, \ell_4), \tag{5.12}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_{\text{LSS}}^{(22)}(\ell) &= \frac{1}{2\pi} \int d^2x \frac{1}{2} \left[ \theta_{\text{LSS}}^{a(2)} \theta_{\text{LSS}}^{b(2)} + 2\theta^{a(2)} \theta_{\text{LSS}}^{b(2)} \right] \nabla_a \nabla_b \mathcal{M} e^{i\ell \cdot \mathbf{x}} \\
&= -\frac{1}{2} \frac{1}{\pi^2} \int d^2\ell_2 \int d^2\ell_3 [(\ell + \ell_2 - \ell_3) \cdot \ell_3] (\ell_2 \cdot \ell_3) \\
&\quad \times \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^{r_s} dr' \frac{r_s - r'}{r_s r'} \Phi_W^{(2)}(r, \ell + \ell_2 - \ell_3) \bar{\Phi}_W^{(2)}(r', \ell_2) \mathcal{M}(r_s, \ell_3) \\
&\quad + \frac{1}{\pi^3} \int d^2\ell_2 \int d^2\ell_3 \int d^2\ell_4 [(\ell - \ell_2 - \ell_3 - \ell_4) \cdot \ell_4] (\ell_4 \cdot \ell_2) (\ell_3 \cdot \ell_2) \\
&\quad \times \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^{r_s} dr' \frac{r_s - r'}{r_s r'} \int_0^{r'} dr'' \frac{r' - r''}{r' r''} \\
&\quad \times \Phi_W^{(2)}(r, \ell - \ell_2 - \ell_3 - \ell_4) \Phi_W(r', \ell_2) \Phi_W(r'', \ell_3) \mathcal{M}(r_s, \ell_4). \tag{5.13}
\end{aligned}$$

With these results and using also the  $\mathcal{A}^{(i, \dots)}(\ell)$  containing only the first order Weyl potential given in [18], we can now determine the LSS contribution to the second group by following the procedure outlined in [18]. We first introduce

$$\begin{aligned}
\langle \Phi_W^{(2)}(z, \ell) \bar{\Phi}_W^{(2)}(z', \ell') \rangle &= \delta(\ell - \ell') C_\ell^{W(22)}(z, z'), \\
\langle \Phi_W(z, \ell) \bar{\Phi}_W^{(3)}(z', \ell') \rangle &= \delta(\ell - \ell') C_\ell^{W(13)}(z, z'), \tag{5.14}
\end{aligned}$$

and

$$\begin{aligned}
C_\ell^{\psi(22)}(z, z') &= 4 \int_0^r dr_1 \frac{r - r_1}{rr_1} \int_0^{r'} dr_2 \frac{r' - r_2}{r'r_2} C_\ell^{W(22)}(z_1, z_2), \\
C_\ell^{\psi(13)}(z, z') &= 4 \int_0^r dr_1 \frac{r - r_1}{rr_1} \int_0^{r'} dr_2 \frac{r' - r_2}{r'r_2} C_\ell^{W(13)}(z_1, z_2).
\end{aligned} \tag{5.15}$$

With this we obtain

$$\begin{aligned}
C_{\ell, LSS}^{(0,22)} + C_{\ell, LSS}^{(0,13)} &= -C_\ell^{\mathcal{M}}(z_s) \int \frac{d^2 \ell_1}{(2\pi)^2} (\ell_1 \cdot \ell)^2 \left[ C_{\ell_1}^{\psi(22)}(z_s, z_s) + 2C_{\ell_1}^{\psi(13)}(z_s, z_s) \right] \\
&\quad - 16C_\ell^{\mathcal{M}}(z_s) \int \frac{d^2 \ell_1}{(2\pi)^2} \int \frac{d^2 \ell_2}{(2\pi)^2} [(\ell_2 + \ell_3) \cdot \ell] (\ell_2 \cdot \ell) (\ell_3 \cdot \ell) \\
&\quad \times \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^{r_s} dr' \frac{r_s - r'}{r_s r'} \int_0^{r'} dr'' \frac{r' - r''}{r' r''} \\
&\quad \times b_{|\ell_2 + \ell_3| \ell_2 \ell_3}^{\Phi \Phi \Phi^{(2)}}(r, r', r''),
\end{aligned} \tag{5.16}$$

$$\begin{aligned}
C_{\ell, LSS}^{(2,2)} + C_{\ell, LSS}^{(1,3)} &= \int \frac{d^2 \ell_1}{(2\pi)^2} [(\ell - \ell_1) \cdot \ell_1]^2 \left[ C_{|\ell - \ell_1|}^{\psi(22)}(z_s, z_s) + C_{|\ell - \ell_1|}^{\psi(13)}(z_s, z_s) \right] C_{\ell_1}^{\mathcal{M}}(z_s) \\
&\quad - 16 \int \frac{d^2 \ell_1}{(2\pi)^2} \int \frac{d^2 \ell_2}{(2\pi)^2} [(\ell + \ell_2 - \ell_1) \cdot \ell_2] [(\ell - \ell_1) \cdot \ell_1] [(\ell + \ell_2 - \ell_1) \cdot \ell_1] \\
&\quad \times C_{\ell_1}^{\mathcal{M}}(z_s) \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^{r_s} dr' \frac{r_s - r'}{r_s r'} \int_0^{r'} dr'' \frac{r' - r''}{r' r''} \\
&\quad \times b_{|\ell - \ell_1| |\ell - \ell_1 + \ell_2| \ell_2}^{\Phi \Phi \Phi^{(2)}}(r, r', r''),
\end{aligned} \tag{5.17}$$

where  $b_{\ell_1 \ell_2 \ell_3}^{\Phi \Phi \Phi^{(2)}}$  is a reduced bispectrum and is defined by

$$\langle \Phi_W^{(2)}(r_1, \ell_1) \Phi_W(r_2, \ell_2) \Phi_W(r_3, \ell_3) \rangle_c + \text{perm.} = \delta_D(\ell_1 + \ell_2 + \ell_3) \frac{1}{2\pi} b_{\ell_1 \ell_2 \ell_3}^{\Phi^{(2)} \Phi \Phi}(r_1, r_2, r_3). \tag{5.18}$$

Following Sec. 3.4 of [37] and using the Limber approximation we obtain the following expression for the reduced bispectrum

$$\begin{aligned}
b_{\ell_1 \ell_2 \ell_3}^{\Phi^{(2)} \Phi \Phi}(z_1, z_2, z_3) &= -\frac{1}{12} [\mathcal{H}(\eta_1)^2 (\Omega_m(\eta_1))]^{-1} \frac{\delta_D(r_2 - r_3) \delta_D(r_1 - r_3)}{r_3^2} \nu_2^2 \nu_3^2 \frac{1}{(\ell_1 + 1/2)^2} \\
&\quad P_R(\nu_2) P_R(\nu_3) T_{\Phi+\Psi}^2(\nu_2, \eta_3) T_{\Phi+\Psi}^2(\nu_3, \eta_3) F_2 \left( \frac{\ell_1 + 1/2}{r_3}, \nu_2, \nu_3 \right) \\
&\quad + \text{perm.},
\end{aligned} \tag{5.19}$$

where  $\nu_i \equiv \frac{\ell_i + 1/2}{r_i}$ ,  $r_i = r(z_i)$  as well as  $\eta_i = \eta(z_i)$  and we define (see [37])

$$F_2(k_1, k_2, k_3) = \frac{5}{7} + \frac{1}{4} \frac{k_1^2 - k_2^2 - k_3^2}{k_2 k_3} \left( \frac{k_2}{k_3} + \frac{k_3}{k_2} \right) + \frac{1}{14} \left( \frac{k_1^2 - k_2^2 - k_3^2}{k_2 k_3} \right)^2. \tag{5.20}$$

The first contributions to Eqs. (5.16) and (5.17) take care of when we take into account higher order contributions to the gravitational potential in  $C_\ell^\psi$  (a higher order power

spectrum) and, therefore, it is included when we consider a Halofit model in evaluating the leading first order contribution (in the sense that if we add this contribution to the first order contribution evaluated via Halofit we would effectively do a double counting). The second terms in Eqs. (5.16) and (5.17), depend on the reduced bispectrum. In the Limber approximation given in Eq. (5.19) these contributions vanish due to the Dirac-delta function,  $\delta(r' - r'')$ .

## 5.2 Third group

We now evaluate the LSS corrections to our third group. In this group no 3rd order perturbation occur and it is sufficient to consider the LSS contribution in the deflection angle up to second order.

From the definitions in Eqs. (3.20) and (3.26) the LSS contribution to our third group is due to the contribution of  $\Phi_W^{(2)}$  present in  $\mathcal{A}^{(2)}(\ell)$  and  $\mathcal{A}^{(12)}(\ell)$ . The expression for  $\mathcal{A}_{\text{LSS}}^{(2)}(\ell)$  is given in Eq. (5.10). While, following [18] we obtain

$$\begin{aligned}\mathcal{A}_{\text{LSS}}^{(12)}(\ell) &= \frac{1}{2\pi} \int d^2x \theta^{a(1)} \theta_{\text{LSS}}^{b(2)} \nabla_a \nabla_b \mathcal{M} e^{i\ell \cdot \mathbf{x}} \\ &= -\frac{1}{\pi^2} \int d^2\ell_2 \int d^2\ell_3 [(\ell + \ell_2 - \ell_3) \cdot \ell_3] (\ell_2 \cdot \ell_3) \int_0^{r_s} dr \frac{r_s - r}{r_s r} \\ &\quad \times \int_0^{r_s} dr' \frac{r_s - r'}{r_s r'} \Phi_W(r, \ell + \ell_2 - \ell_3) \bar{\Phi}_W^{(2)}(r', \ell_2) \mathcal{M}(r_s, \ell_3). \end{aligned} \quad (5.21)$$

Using Eqs. (5.10) and (5.21), the expression for  $\mathcal{A}^{(1)}(\ell)$  and  $\mathcal{A}^{(11)}(\ell)$  given in [18], and Eq. (3.20), we then obtain the following LSS contribution to the third group

$$\begin{aligned}C_{\ell, \text{LSS}}^{(1,12)} + C_{\ell, \text{LSS}}^{(2,11)} &= -8 \int \frac{d^2\ell_1}{(2\pi)^2} \int \frac{d^2\ell_2}{(2\pi)^2} (\ell_1 \cdot \ell_2) [(\ell - \ell_1) \cdot \ell_1] [(\ell + \ell_2 - \ell_1) \cdot \ell_1] \\ &\quad \times C_{\ell_1}^{\mathcal{M}}(z_s) \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^{r_s} dr' \frac{r_s - r'}{r_s r'} \int_0^{r_s} dr'' \frac{r_s - r''}{r_s r''} \\ &\quad \times b_{|\ell - \ell_1| |\ell - \ell_1 + \ell_2| \ell_2}^{\Phi^{(2)} \Phi \Phi}(r, r', r''). \end{aligned} \quad (5.22)$$

Note that this result remains finite in the Limber approximation for the reduced bispectrum as there is no factor  $r' - r''$  in the integrand. Our expression (5.22) for the LSS correction agrees with the corresponding result of Ref. [23].

## 6 Contribution from rotation

When considering the next-to-leading order corrections to the CMB polarization, another new effect needs to be taken into account: polarization is oriented along a given direction at emission and this direction may rotate along the path of the photon to the observer position. More precisely, introducing the rotation angle  $\beta$ , the effect of this rotation on Eq. (2.4) is given by a rotation matrix  $\mathcal{R}_A^B$  (see Eq. (C.8)) acting on the Sachs basis, as defined in Appendix C. To evaluate it, the polarization tensor  $\mathcal{P}_{mn}$  is projected on a screen at the observer position

given by Eq. (C.11) which is rotated by an angle  $\beta$  with respect to the parallel transported screen at the source. Because the screen basis vectors appear twice in the projection of the polarization tensor, a rotation on it will change  $\mathcal{P}$  by  $2\beta$ . This is simply a consequence of the spin-2 nature of the polarization tensor. Starting from [38, 39]

$$\tilde{\mathcal{P}}^{mn}(x^a) 2\tilde{s}_m^{(+)}\tilde{s}_n^{(+)} = \mathcal{P}^{mn}(x^a + \delta\theta^a) 2\tilde{s}_m^{(+)}\tilde{s}_n^{(+)}, \quad (6.1)$$

with  $\tilde{s}_m^{(+)} = e^{-i\beta} s_m^{(+)}$  and  $s_m^{(\pm)} = \frac{1}{\sqrt{2}}(s_m^1 \pm i s_m^2)$ , we obtain

$$\tilde{\mathcal{P}}(x^a) = e^{-2i\beta} \mathcal{P}(x^a + \delta\theta^a). \quad (6.2)$$

Because we are interested in next-to-leading order corrections, we have to take into account the expansion of  $\beta$  up to fourth order,  $\beta \simeq \beta^{(0)} + \beta^{(1)} + \beta^{(2)} + \beta^{(3)} + \beta^{(4)}$ . As explained in [38, 39], in their framework this angle is equivalent to the angle  $\omega$  determined by the antisymmetric part of the amplification matrix. Qualitatively,  $\omega$  and  $\beta$  refer to different physical rotations: the vorticity  $\omega$  takes into account the rotation of a bundle of light rays which travel together, whereas  $\beta$  is meaningful also just for a single photon. Nevertheless, in Appendix C we show that these angles are equal also for scalar fluctuations and they are both sourced by the curl potential  $\Omega$  in the amplification matrix  $\Psi_b^a$  (see [18] for definitions). More precisely,

$$\beta = -\frac{1}{2}\Delta\Omega \quad (6.3)$$

which is exactly the vorticity  $\omega$ . In [39] this equality has been proved for first order vector and tensor perturbations. In Appendix C we calculate  $\beta$  due to scalar perturbations without reference to the amplification matrix, by directly solving the parallel transport equation for the Sachs basis, and show the equality  $\omega = \beta$  up to second order. Indeed, we find that  $\beta^{(0)}$  and  $\beta^{(1)}$  are constant along the geodesic, so there is no rotation of polarization between source and observer up to first order. With a global rotation of the Sachs basis we can achieve  $\beta^{(0)} = \beta^{(1)} = 0$ . This is perfectly consistent with Eq. (6.3) since also  $\omega^{(0)} = \omega^{(1)} = 0$  for purely scalar first order perturbations. Moreover, a non trivial evaluation shows explicitly that  $\beta^{(2)} = \omega^{(2)}$  (see Eq. (C.30) and its derivation in Appendix C for details).

In principle, we should take into account also  $\beta^{(3)}$  and  $\beta^{(4)}$ . However, because of the structure of the rotation, we can neglect all the terms which contain only one angle  $\beta^{(i)}$  (this is again a consequence of statistical isotropy). Now, the fact that  $\beta^{(0)} = \beta^{(1)} = 0$  implies that  $\beta^{(3)}$  and  $\beta^{(4)}$  can only appear alone in the spectra, so they do not contribute at next-to-leading order.



In general, the full expansion of the polarization up to 4th order reads

$$\begin{aligned}
\tilde{\mathcal{P}}(x^a) &= e^{-2i(\beta^{(2)}+\beta^{(3)}+\beta^{(4)})} \mathcal{P}(x^a + \theta^{a(1)} + \theta^{a(2)} + \theta^{a(3)}) \\
&\simeq \left[ 1 - 2i\beta^{(2)} - 2i\beta^{(3)} - 2i\beta^{(4)} - 2\left(\beta^{(2)}\right)^2 \right] \\
&\quad \times \left[ \mathcal{D}^{(0)}(x^a) + \sum_{i=1}^4 \mathcal{D}^{(i)}(x^a) + \sum_{\substack{i+j \leq 4 \\ 1 \leq i \leq j}} \mathcal{D}^{(ij)}(x^a) + \sum_{\substack{i+j+k \leq 4 \\ 1 \leq i \leq j \leq k}} \mathcal{D}^{(ijk)}(x^a) + \mathcal{D}^{(1111)}(x^a) \right] \\
&\simeq \mathcal{D}^{(0)}(x^a) + \sum_{i=1}^4 \mathcal{D}^{(i)}(x^a) + \sum_{\substack{i+j \leq 4 \\ 1 \leq i \leq j}} \mathcal{D}^{(ij)}(x^a) + \sum_{\substack{i+j+k \leq 4 \\ 1 \leq i \leq j \leq k}} \mathcal{D}^{(ijk)}(x^a) + \mathcal{D}^{(1111)}(x^a) \\
&\quad - 2i\beta^{(2)} \left[ \mathcal{D}^{(0)}(x^a) + \sum_{i=1}^2 \mathcal{D}^{(i)}(x^a) + \mathcal{D}^{(11)}(x^a) \right] \\
&\quad - 2i\beta^{(3)} \left[ \mathcal{D}^{(0)}(x^a) + \mathcal{D}^{(1)}(x^a) \right] - \left[ 2i\beta^{(4)} + 2\left(\beta^{(2)}\right)^2 \right] \mathcal{D}^{(0)}(x^a). \tag{6.4}
\end{aligned}$$

According to what we explained above, only two more terms containing  $\beta^{(2)}$  contribute, namely

$$-2i\beta^{(2)}\mathcal{D}^{(0)} \quad \text{and} \quad -2\left(\beta^{(2)}\right)^2\mathcal{D}^{(0)}. \tag{6.5}$$

Expressing the result for  $\beta^{(2)}$  given in Appendix C, in  $\ell$  space, we obtain

$$\begin{aligned}
\mathcal{R}^{(2)}(\ell) &= -\frac{2i}{2\pi} \int d^2x \beta^{(2)} \mathcal{D}^{(0)} e^{i\ell \cdot x} \\
&= -\frac{4i}{(2\pi)^2} \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^r dr_1 \frac{r - r_1}{r r_1} \int d^2\ell_1 \int d^2\ell_2 \\
&\quad \times [\mathbf{n} \cdot (\ell_2 \wedge \ell_1) (\ell_1 \cdot \ell_2)] \Phi_W(z, \ell_1) \Phi_W(z_1, \ell_2) \mathcal{D}^{(0)}(\ell - \ell_1 - \ell_2), \tag{6.6}
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}^{(22)}(\ell) &= -\frac{2}{2\pi} \int d^2x \left(\beta^{(2)}\right)^2 \mathcal{D}^{(0)} e^{i\ell \cdot x} \\
&= -\frac{8}{(2\pi)^4} \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^r dr_1 \frac{r - r_1}{r r_1} \\
&\quad \times \int_0^{r_s} dr_2 \frac{r_s - r_2}{r_s r_2} \int_0^{r_2} dr_3 \frac{r_2 - r_3}{r_2 r_3} \int d^2\ell_2 \int d^2\ell_3 \int d^2\ell_4 \int d^2\ell_5 \\
&\quad \times [\mathbf{n} \cdot (\ell_2 \wedge \ell_1) (\ell_1 \cdot \ell_2)] [\mathbf{n} \cdot (\ell_4 \wedge \ell_3) (\ell_3 \cdot \ell_4)] \\
&\quad \times \Phi_W(z, \ell_1) \Phi_W(z_1, \ell_2) \Phi_W(z_2, \ell_3) \Phi_W(z_3, \ell_4) \mathcal{D}^{(0)}(z_s, \ell - \ell_1 - \ell_2 - \ell_3 - \ell_4). \tag{6.7}
\end{aligned}$$

Here, as in Appendix C,  $\mathbf{n}$  is the unit vector normal to the  $\ell$ -plane. Using these expansions, we can now evaluate the contribution of  $\beta^{(2)}$  to polarization. The new non-vanishing terms

are (see Appendix B for similar calculation for post-Born and LSS contributions)

$$\begin{aligned}
\delta(\ell - \ell') \Delta (C_\ell^\mathcal{E} + C_\ell^\mathcal{B})^{(22,0)} &= \langle \mathcal{R}^{(22)}(\ell) \bar{\mathcal{D}}^{(0)}(\ell') \rangle, \\
\delta(\ell - \ell') \Delta (C_\ell^\mathcal{E} + C_\ell^\mathcal{B})^{(2,2)} &= \langle \mathcal{R}^{(2)}(\ell) \bar{\mathcal{R}}^{(2)}(\ell') \rangle, \\
e^{-4i\phi_\ell} \delta(\ell + \ell') \Delta (C_\ell^\mathcal{E} - C_\ell^\mathcal{B})^{(22,0)} &= \langle \mathcal{R}^{(22)}(\ell) \mathcal{D}^{(0)}(\ell') \rangle, \\
e^{-4i\phi_\ell} \delta(\ell + \ell') \Delta (C_\ell^\mathcal{E} - C_\ell^\mathcal{B})^{(2,2)} &= \langle \mathcal{R}^{(2)}(\ell) \mathcal{R}^{(2)}(\ell') \rangle, \\
-e^{-2i\phi_\ell} \delta(\ell - \ell') \Delta C_\ell^{\mathcal{EM}(22,0)} &= \langle \mathcal{R}^{(22)}(\ell) \bar{\mathcal{A}}^{(0)}(\ell') \rangle.
\end{aligned} \tag{6.8}$$

Inserting our expressions for  $\mathcal{R}^{(22)}$ ,  $\mathcal{R}^{(2)}$ ,  $\mathcal{D}^{(0)}$  and  $\mathcal{A}^{(0)}$  we find

$$\begin{aligned}
\Delta (C_\ell^\mathcal{E} + C_\ell^\mathcal{B})^{(22,0)} &= -8 [C_\ell^\mathcal{E}(z_s) + C_\ell^\mathcal{B}(z_s)] \int \frac{d^2\ell_1}{(2\pi)^2} \int \frac{d^2\ell_2}{(2\pi)^2} [\mathbf{n} \cdot (\ell_2 \wedge \ell_1) (\ell_1 \cdot \ell_2)]^2 \\
&\times \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^r dr_1 \frac{r - r_1}{r r_1} \int_0^{r_s} dr_2 \frac{r_s - r_2}{r_s r_2} \int_0^{r_2} dr_3 \frac{r_2 - r_3}{r_2 r_3} \\
&\times [C_{\ell_1}^W(z, z_2) C_{\ell_2}^W(z_1, z_3) - C_{\ell_1}^W(z, z_3) C_{\ell_2}^W(z_1, z_2)] ,
\end{aligned} \tag{6.9}$$

$$\begin{aligned}
\Delta (C_\ell^\mathcal{E} - C_\ell^\mathcal{B})^{(22,0)} &= -8 [C_\ell^\mathcal{E}(z_s) - C_\ell^\mathcal{B}(z_s)] \int \frac{d^2\ell_1}{(2\pi)^2} \int \frac{d^2\ell_2}{(2\pi)^2} [\mathbf{n} \cdot (\ell_2 \wedge \ell_1) (\ell_1 \cdot \ell_2)]^2 \\
&\times \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^r dr_1 \frac{r - r_1}{r r_1} \int_0^{r_s} dr_2 \frac{r_s - r_2}{r_s r_2} \int_0^{r_2} dr_3 \frac{r_2 - r_3}{r_2 r_3} \\
&\times [C_{\ell_1}^W(z, z_2) C_{\ell_2}^W(z_1, z_3) - C_{\ell_1}^W(z, z_3) C_{\ell_2}^W(z_1, z_2)] ,
\end{aligned} \tag{6.10}$$

$$\begin{aligned}
\Delta (C_\ell^\mathcal{E} + C_\ell^\mathcal{B})^{(2,2)} &= 16 \int \frac{d^2\ell_1}{(2\pi)^2} \int \frac{d^2\ell_2}{(2\pi)^2} [\mathbf{n} \cdot (\ell_2 \wedge \ell_1) (\ell_1 \cdot \ell_2)]^2 \\
&\times \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^r dr_1 \frac{r - r_1}{r r_1} \int_0^{r_s} dr_2 \frac{r_s - r_2}{r_s r_2} \int_0^{r_2} dr_3 \frac{r_2 - r_3}{r_2 r_3} \\
&\times [C_{|\ell - \ell_1 - \ell_2|}^\mathcal{E}(z_s) + C_{|\ell - \ell_1 - \ell_2|}^\mathcal{B}(z_s)] \\
&\times [C_{\ell_1}^W(z, z_2) C_{\ell_2}^W(z_1, z_3) - C_{\ell_1}^W(z, z_3) C_{\ell_2}^W(z_1, z_2)] ,
\end{aligned} \tag{6.11}$$

$$\begin{aligned}
\Delta (C_\ell^\mathcal{E} - C_\ell^\mathcal{B})^{(2,2)} &= -16 \int \frac{d^2\ell_1}{(2\pi)^2} \int \frac{d^2\ell_2}{(2\pi)^2} [\mathbf{n} \cdot (\ell_2 \wedge \ell_1) (\ell_1 \cdot \ell_2)]^2 \\
&\times \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^r dr_1 \frac{r - r_1}{r r_1} \int_0^{r_s} dr_2 \frac{r_s - r_2}{r_s r_2} \int_0^{r_2} dr_3 \frac{r_2 - r_3}{r_2 r_3} \\
&\times [C_{|\ell - \ell_1 - \ell_2|}^\mathcal{E}(z_s) - C_{|\ell - \ell_1 - \ell_2|}^\mathcal{B}(z_s)] \\
&\times \{ \cos^2 [2(\phi_\ell - \phi_{|\ell - \ell_1 - \ell_2|})] - \sin^2 [2(\phi_\ell - \phi_{|\ell - \ell_1 - \ell_2|})] \} \\
&\times [C_{\ell_1}^W(z, z_2) C_{\ell_2}^W(z_1, z_3) - C_{\ell_1}^W(z, z_3) C_{\ell_2}^W(z_1, z_2)] ,
\end{aligned} \tag{6.12}$$

$$\begin{aligned}
\Delta C_\ell^{\mathcal{EM}(22,0)} &= -8 C_\ell^{\mathcal{EM}}(z_s) \int \frac{d^2 \ell_1}{(2\pi)^2} \int \frac{d^2 \ell_2}{(2\pi)^2} [\mathbf{n} \cdot (\ell_2 \wedge \ell_1) (\ell_1 \cdot \ell_2)]^2 \\
&\times \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^r dr_1 \frac{r - r_1}{r r_1} \int_0^{r_s} dr_2 \frac{r_s - r_2}{r_s r_2} \int_0^{r_2} dr_3 \frac{r_2 - r_3}{r_2 r_3} \\
&\times [C_{\ell_1}^W(z, z_2) C_{\ell_2}^W(z_1, z_3) - C_{\ell_1}^W(z, z_3) C_{\ell_2}^W(z_1, z_2)] .
\end{aligned} \tag{6.13}$$

From  $\Delta (C_\ell^{\mathcal{E}} \pm C_\ell^{\mathcal{B}})$ , we can easily obtain the corrections to  $C_\ell^{\mathcal{E}}$  and  $C_\ell^{\mathcal{B}}$ ,

$$\begin{aligned}
\Delta C_\ell^{\mathcal{E}(22,0)} &\equiv \frac{1}{2} \left[ \Delta (C_\ell^{\mathcal{E}} + C_\ell^{\mathcal{B}})^{(22,0)} + \Delta (C_\ell^{\mathcal{E}} - C_\ell^{\mathcal{B}})^{(22,0)} \right] \\
&= -8 C_\ell^{\mathcal{E}}(z_s) \int \frac{d^2 \ell_1}{(2\pi)^2} \int \frac{d^2 \ell_2}{(2\pi)^2} [\mathbf{n} \cdot (\ell_2 \wedge \ell_1) (\ell_1 \cdot \ell_2)]^2 \\
&\times \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^r dr_1 \frac{r - r_1}{r r_1} \int_0^{r_s} dr_2 \frac{r_s - r_2}{r_s r_2} \int_0^{r_2} dr_3 \frac{r_2 - r_3}{r_2 r_3} \\
&\times [C_{\ell_1}^W(z, z_2) C_{\ell_2}^W(z_1, z_3) - C_{\ell_1}^W(z, z_3) C_{\ell_2}^W(z_1, z_2)] ,
\end{aligned} \tag{6.14}$$

$$\begin{aligned}
\Delta C_\ell^{\mathcal{E}(2,2)} &\equiv \frac{1}{2} \left[ \Delta (C_\ell^{\mathcal{E}} + C_\ell^{\mathcal{B}})^{(2,2)} + \Delta (C_\ell^{\mathcal{E}} - C_\ell^{\mathcal{B}})^{(2,2)} \right] \\
&= 16 \int \frac{d^2 \ell_1}{(2\pi)^2} \int \frac{d^2 \ell_2}{(2\pi)^2} [\mathbf{n} \cdot (\ell_2 \wedge \ell_1) (\ell_1 \cdot \ell_2)]^2 \\
&\times \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^r dr_1 \frac{r - r_1}{r r_1} \int_0^{r_s} dr_2 \frac{r_s - r_2}{r_s r_2} \int_0^{r_2} dr_3 \frac{r_2 - r_3}{r_2 r_3} \\
&\times [C_{\ell_1}^W(z, z_2) C_{\ell_2}^W(z_1, z_3) - C_{\ell_1}^W(z, z_3) C_{\ell_2}^W(z_1, z_2)] \\
&\times \left\{ C_{|\ell - \ell_1 - \ell_2|}^{\mathcal{E}}(z_s) \sin^2 [2 (\phi_\ell - \phi_{|\ell - \ell_1 - \ell_2|})] \right. \\
&\left. + C_{|\ell - \ell_1 - \ell_2|}^{\mathcal{B}}(z_s) \cos^2 [2 (\phi_\ell - \phi_{|\ell - \ell_1 - \ell_2|})] \right\} ,
\end{aligned} \tag{6.15}$$

$$\begin{aligned}
\Delta C_\ell^{\mathcal{B}(22,0)} &\equiv \frac{1}{2} \left[ \Delta (C_\ell^{\mathcal{E}} + C_\ell^{\mathcal{B}})^{(22,0)} - \Delta (C_\ell^{\mathcal{E}} - C_\ell^{\mathcal{B}})^{(22,0)} \right] \\
&= -8 C_\ell^{\mathcal{B}}(z_s) \int \frac{d^2 \ell_1}{(2\pi)^2} \int \frac{d^2 \ell_2}{(2\pi)^2} [\mathbf{n} \cdot (\ell_2 \wedge \ell_1) (\ell_1 \cdot \ell_2)]^2 \\
&\times \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^r dr_1 \frac{r - r_1}{r r_1} \int_0^{r_s} dr_2 \frac{r_s - r_2}{r_s r_2} \int_0^{r_2} dr_3 \frac{r_2 - r_3}{r_2 r_3} \\
&\times [C_{\ell_1}^W(z, z_2) C_{\ell_2}^W(z_1, z_3) - C_{\ell_1}^W(z, z_3) C_{\ell_2}^W(z_1, z_2)] ,
\end{aligned} \tag{6.16}$$

$$\begin{aligned}
\Delta C_\ell^{\mathcal{B}(2,2)} &\equiv \frac{1}{2} \left[ \Delta (C_\ell^{\mathcal{E}} + C_\ell^{\mathcal{B}})^{(2,2)} - \Delta (C_\ell^{\mathcal{E}} - C_\ell^{\mathcal{B}})^{(2,2)} \right] \\
&= 16 \int \frac{d^2 \ell_1}{(2\pi)^2} \int \frac{d^2 \ell_2}{(2\pi)^2} [\mathbf{n} \cdot (\ell_2 \wedge \ell_1) (\ell_1 \cdot \ell_2)]^2 \\
&\times \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^r dr_1 \frac{r - r_1}{r r_1} \int_0^{r_s} dr_2 \frac{r_s - r_2}{r_s r_2} \int_0^{r_2} dr_3 \frac{r_2 - r_3}{r_2 r_3} \\
&\times [C_{\ell_1}^W(z, z_2) C_{\ell_2}^W(z_1, z_3) - C_{\ell_1}^W(z, z_3) C_{\ell_2}^W(z_1, z_2)] \\
&\times \left\{ C_{|\ell - \ell_1 - \ell_2|}^{\mathcal{E}}(z_s) \cos^2 [2 (\phi_\ell - \phi_{|\ell - \ell_1 - \ell_2|})] \right. \\
&\left. + C_{|\ell - \ell_1 - \ell_2|}^{\mathcal{B}}(z_s) \sin^2 [2 (\phi_\ell - \phi_{|\ell - \ell_1 - \ell_2|})] \right\} .
\end{aligned} \tag{6.17}$$

In a final step we apply the Limber approximation to our integrals. We note that we always encounter the same time integrals, therefore we can evaluate this approximation once and then apply it to all our terms. Within the Limber approximation, the  $C_\ell$ 's for the Weyl potential become

$$C_{\ell_1}^W(z, z_2)C_{\ell_2}^W(z_1, z_3) - C_{\ell_1}^W(z, z_3)C_{\ell_2}^W(z_1, z_2) = \frac{\delta(r_2 - r)\delta(r_3 - r_1) - \delta(r_3 - r)\delta(r_2 - r_1)}{16 r^2 r_1^2} \\ \times P_R\left(\frac{\ell_1 + 1/2}{r}\right) \left[T_{\Phi+\Psi}\left(\frac{\ell_1 + 1/2}{r}, z\right)\right]^2 P_R\left(\frac{\ell_2 + 1/2}{r_1}\right) \left[T_{\Phi+\Psi}\left(\frac{\ell_2 + 1/2}{r_1}, z_1\right)\right]^2, \quad (6.18)$$

so that

$$\int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^r dr_1 \frac{r - r_1}{r r_1} \int_0^{r_s} dr_2 \frac{r_s - r_2}{r_s r_2} \int_0^{r_2} dr_3 \frac{r_2 - r_3}{r_2 r_3} \\ \times [C_{\ell_1}^W(z, z_2)C_{\ell_2}^W(z_1, z_3) - C_{\ell_1}^W(z, z_3)C_{\ell_2}^W(z_1, z_2)] \\ = \frac{1}{16} \int_0^{r_s} \frac{dr}{r^2} \int_0^r \frac{dr_1}{r_1^2} \left(\frac{r - r_1}{r r_1}\right)^2 \left(\frac{r_s - r}{r_s r}\right)^2 P_R\left(\frac{\ell_1 + 1/2}{r}\right) \\ \times P_R\left(\frac{\ell_2 + 1/2}{r_1}\right) \left[T_{\Phi+\Psi}\left(\frac{\ell_1 + 1/2}{r}, z\right)\right]^2 \left[T_{\Phi+\Psi}\left(\frac{\ell_2 + 1/2}{r_1}, z_1\right)\right]^2. \quad (6.19)$$

This simplification applies to all the contributions evaluated above.

## 7 Numerical Results

In this section we present the numerical evaluation of the results given above. For the numerical results we consider non-linear (Halofit model [19, 20]) power spectra for the gravitational potential. All the figures have been generated with the following cosmological parameters  $h = 0.67$ ,  $\omega_{\text{cdm}} = 0.12$ ,  $\omega_b = 0.022$  and vanishing curvature. The primordial curvature power spectrum has the amplitude  $A_s = 2.215 \times 10^{-9}$ , the pivot scale  $k_{\text{pivot}} = 0.05 \text{ Mpc}^{-1}$ , the spectral index  $n_s = 0.96$  and no running. The transfer function for the Bardeen potentials,  $T_{\Phi+\Psi}$  has been computed with CLASS [16], using Halofit [20]. In analysing the contribution of  $R_{\beta(2)}$  (see below) we compare the non-linear and the linear results. The latter has been obtained with the same cosmological parameters with the linear power spectrum computed with CLASS [16].

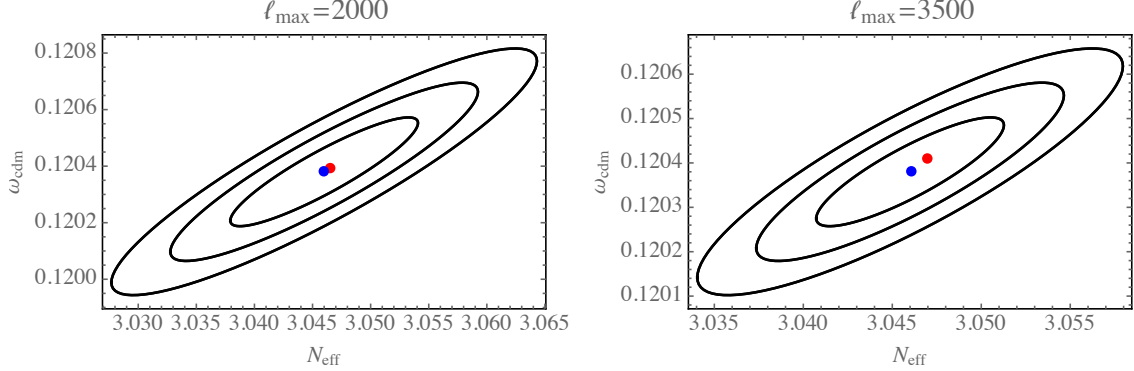
First of all, let us note that all the contributions  $\Delta C_\ell^{X(22,0)}$  from the rotation of polarization contain the same constant factor multiplying simply the unperturbed spectrum. Let us call it  $\mathcal{R}_{\beta(2)}$ , so we have that

$$\frac{\Delta C_\ell^{\mathcal{E}(22,0)}}{C_\ell^{\mathcal{E}}} = \frac{\Delta C_\ell^{\mathcal{B}(22,0)}}{C_\ell^{\mathcal{B}}} = \frac{\Delta C_\ell^{\mathcal{EM}(22,0)}}{C_\ell^{\mathcal{EM}}} = \mathcal{R}_{\beta(2)} \quad (7.1)$$

with

$$\mathcal{R}_{\beta(2)} = -\frac{1}{16} \int \frac{d\ell_1}{2\pi} \int \frac{d\ell_2}{2\pi} (\ell_1 \ell_2)^5 \int_0^{r_s} \frac{dr}{r^2} \int_0^r \frac{dr_1}{r_1^2} \left(\frac{r - r_1}{r r_1}\right)^2 \left(\frac{r_s - r}{r_s r}\right)^2 P_R\left(\frac{\ell_1 + 1/2}{r}\right) \\ \times P_R\left(\frac{\ell_2 + 1/2}{r_1}\right) \left[T_{\Phi+\Psi}\left(\frac{\ell_1 + 1/2}{r}, z\right)\right]^2 \left[T_{\Phi+\Psi}\left(\frac{\ell_2 + 1/2}{r_1}, z_1\right)\right]^2, \quad (7.2)$$

where we have performed the angular integration. Using the linear power spectrum [16] we obtain  $\mathcal{R}_{\beta^{(2)}}^{\text{lin}} = -7.8 \times 10^{-6}$ , whereas using Halofit [20] for the matter power spectrum the term becomes more than one order of magnitude larger, with  $\mathcal{R}_{\beta^{(2)}}^{\text{Halofit}} = -2.5 \times 10^{-4}$ .



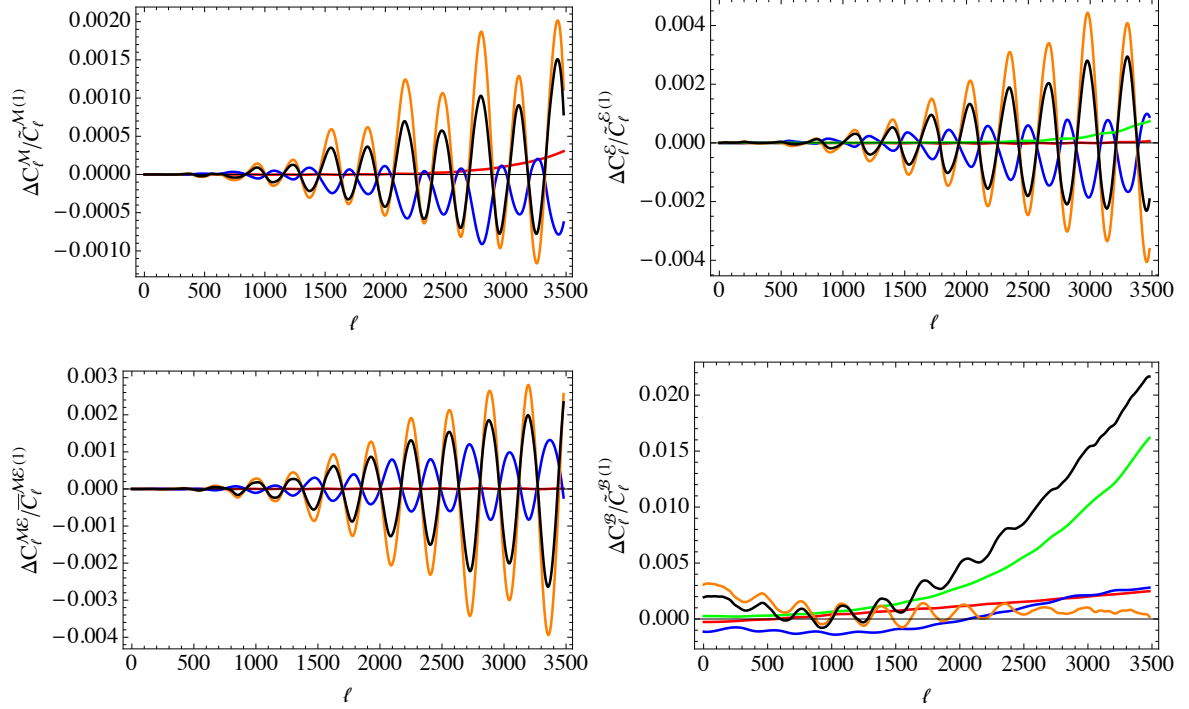
**Figure 1.** Fisher forecast (see Appendix D for details) for a cosmic variance limited survey. The blue (red) points show the shift in the best fit parameter for the dark matter density  $\omega_{\text{cdm}} = h^2 \Omega_{\text{cdm}}$  and the effective number of relativistic species  $N_{\text{eff}}$  induced by the terms in Eqs. (6.13) and (6.14) (we consider vanishing primordial B-modes) using the linear power spectrum (using Halofit). The unshifted best fit value is covered by the blue point. The ellipses denote 1, 2 and 3 sigma contours. The parameters not shown in the panels are fixed to the fiducial cosmology. For both panels we consider B-mode up to  $\ell_{\text{max}} = 1500$  to be consistent with the conservative specifications of CMB-S4 [9].

The universality of this coupling and its independence on  $\ell$  are due to the fact that, in the related correlators in Eqs. (6.8), no derivatives of  $\mathcal{P}$  appear and the two point correlation function of  $\beta^{(2)}$  is evaluated at the same direction. On the other hand, Eqs. (6.15) and (6.17) still have no angular derivatives of  $\mathcal{P}$ , but they involve the two point correlation function of  $\beta^{(2)}$  in two different directions leading to a dependence on  $\ell$  of the corresponding terms.

The integrals over  $\ell_1$  and  $\ell_2$  in  $\mathcal{R}_{\beta^{(2)}}$  converge very slowly and are highly UV sensitive. In particular, a cutoff independent evaluations involves integration domains in  $\ell$  space where perturbation theory is no longer valid, therefore, also numerical results using Halofit are not reliable. Nevertheless, these corrections just leads to an overall shift of  $\Delta C_\ell / C_\ell$ 's and this contribution is negligible in cosmological parameter estimation (see, for instance, Fig. 1). For this reason, we do not consider these terms in what follows.

In Fig. 2 we compare the different higher order contributions. The non-Gaussian (third group) contributions from the post-Born and LSS corrections are dominant for temperature, E-modes and temperature–E-mode cross correlation spectra, whereas they are of the same order of magnitude as post-Born second group for the B-modes. Moreover, the corrections due to rotation are very important for B-modes in a large range of scales (dominant for  $\ell > 1500$ ) and give non negligible corrections to E-modes for  $\ell > 2500$ .

In Fig. 3 we present the ratio between these corrections and cosmic variance  $(\sigma_\ell^X)^2$  given



**Figure 2.** Higher order lensing contributions from the post-Born second group (red curves), post-Born third group (blue curves), LSS (orange curves), and rotation angle  $\beta^{(2)}$  (green curves, contributions (2,2)). Black curves sum up the total correction. We consider the lensing CMB spectra for temperature (top left-panel), E-modes (top right-panel), cross TE spectra (bottom left-panel, where  $\bar{C}_\ell^{\mathcal{ME}(1)} = \sqrt{\frac{(\tilde{C}_\ell^{\mathcal{ME}(1)})^2 + \tilde{C}_\ell^{\mathcal{M}(1)} \tilde{C}_\ell^{\mathcal{E}(1)}}{2}}$ ) and B-modes (bottom right-panel).

by

$$\sigma_\ell^{\mathcal{M}} = \sqrt{\frac{2}{2\ell+1}} C_\ell^{\mathcal{M}}, \quad (7.3)$$

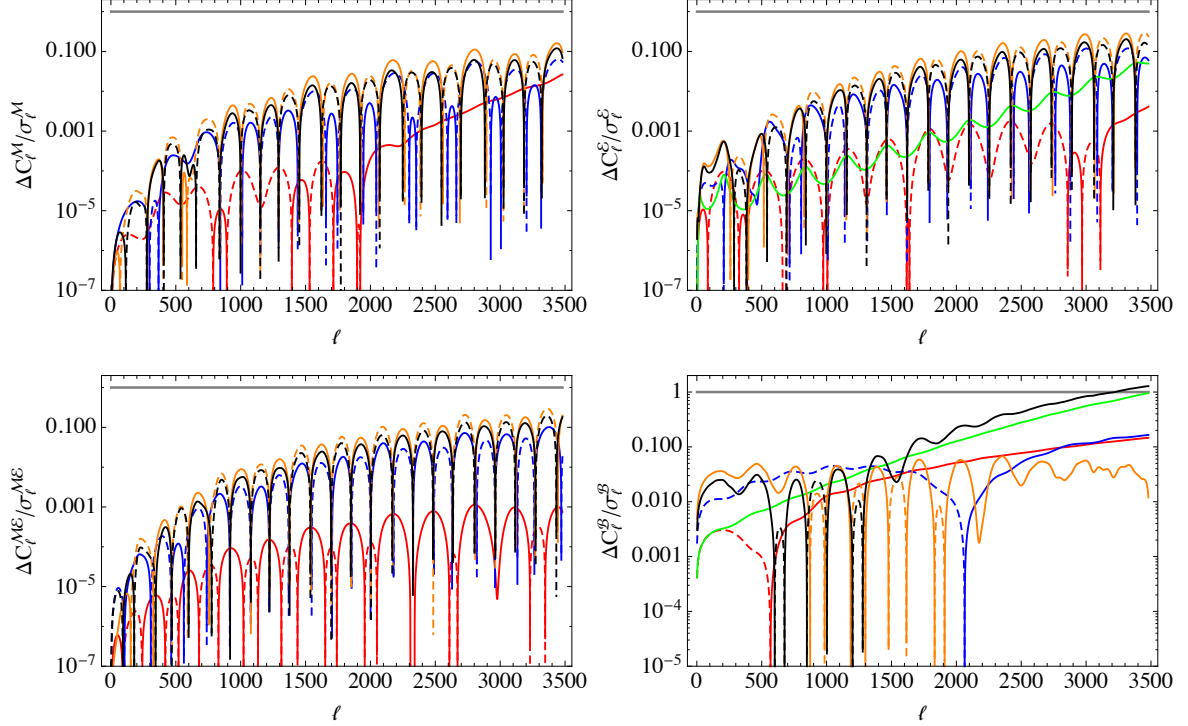
$$\sigma_\ell^{\mathcal{E}} = \sqrt{\frac{2}{2\ell+1}} C_\ell^{\mathcal{E}}, \quad (7.4)$$

$$\sigma_\ell^{\mathcal{ME}} = \sqrt{\frac{1}{2\ell+1}} \sqrt{(C_\ell^{\mathcal{ME}})^2 + C_\ell^{\mathcal{M}} C_\ell^{\mathcal{E}}}, \quad (7.5)$$

$$\sigma_\ell^{\mathcal{B}} = \sqrt{\frac{2}{2\ell+1}} \tilde{C}_\ell^{\mathcal{B}(1)}. \quad (7.6)$$

Note that, for B-modes, we have taken into account the first order resummed correction since we consider no primordial gravitational wave, i.e the unlensed spectrum vanishes. Therefore, lensed B-modes do not have Gaussian statistics. For this reason its cosmic variance can be significantly larger than the one from Eq. (7.6) [40]. Considering Gaussian variance also for B-modes, the corrections due to rotation alone are comparable to cosmic variance for  $\ell \gtrsim 3500$ , in contrast to all other spectra where all the corrections are always below that threshold. Moreover, the sum of all the effects can be even larger than cosmic variance at

these multipoles, showing that higher-order lensing corrections to B-mode polarization at high multipoles have the best chance to be detectable.



**Figure 3.** Comparison between next-to-leading order corrections and cosmic variance for the temperature (Eq. (7.3), top left-panel), E-modes (Eq. (7.4), top right-panel), TE cross correlation (Eq. (7.5), bottom left-panel) and B-modes (Eq. (7.6), bottom right-panel). Red curves refer to post-Born second group, blue curves to post-Born third group, orange to LSS corrections and green curves represent the  $(2,2)$  term of  $\beta^{(2)}$ . Dashed lines are negative values and the black lines trace the sum of all the terms.

## 8 Conclusions

In this paper we have computed all the next-to-leading order corrections to the CMB power spectra of temperature and polarization anisotropies from gravitational lensing of the photons along their path from the last scattering surface into our telescopes. We have found that most terms apart from those already taken into account in present codes [12, 15, 16] are smaller than cosmic variance for a single  $\ell$  mode. The only exception to this rule are the B-mode corrections at very high  $\ell$ . This can be understood from the fact that cosmic variance is proportional to the amplitude of the signal which is by far smallest for the B-modes. Nevertheless by considering the lensed B-modes as Gaussian, we may underestimate their variance [40].

Several of the terms calculated in this paper have already been determined before [18, 22, 23] and our results are in good qualitative agreement, where comparable, with previous

findings. This is a non-trivial consistency check, especially for [22, 23] which use quite different methods. However, the largest correction to the B-modes coming from the rotation of the polarization direction is new. It will be interesting to investigate whether this correction is observable. Let us only note here, that this rotation is due to the vector-degree of freedom of the gravitational field, an effect of frame dragging. Its detection would therefore represent a highly non trivial test of general relativity, testing its elusive spin-1 sector. Recently, it has been proposed to measure this rotation with radio cosmic shear surveys [41].

But also the other terms are not negligible if a precision of 0.1% wants to be achieved as announced in Ref. [17]. For example, for  $\ell$  between 2000 and 2100, cosmic variance amounts to about 2.2%. Hence, as one easily infers from Fig. 3, our corrections with respect to the unlensed spectra are up to the level of 0.1% for the E-polarization spectrum and for the T-E cross correlation, while they are only up to 0.04% for the temperature anisotropy. For the B-polarization spectrum the correction is close to 0.5%.

It is clear that a systematic change even below cosmic variance can affect cosmological parameters and it has to be studied whether next-to-leading order corrections from lensing can indeed influence CMB parameter estimation in the future, this is the topic of an accompanying letter [24]. While, it is unlikely that the tiny corrections of the temperature will be relevant alone, parameters depending strongly on polarization can be affected.

However, independent of parameter estimation, detecting higher order corrections from CMB lensing would be extremely interesting and allow not only a handle on non-linear corrections to the gravitational potential, but also new tests of General Relativity on cosmological scales.

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## A $\mathcal{D}^{(i,\dots)}(\ell)$ terms

In  $\ell$  space, and starting from the result of [18] and of Sect. 5, we obtain the corresponding expressions to evaluate the lensing corrections to the CMB polarization anisotropies up to forth order:

$$\begin{aligned}\mathcal{D}^{(1)}(\ell) &= \frac{1}{2\pi} \int d^2x \theta^{a(1)} \nabla_a \mathcal{P} e^{i\ell \cdot x} \\ &= -\frac{1}{\pi} \int d^2\ell_2 [(\ell - \ell_2) \cdot \ell_2] \int_0^{r_s} dr \frac{r_s - r}{r_s r} \Phi_W(r, \ell - \ell_2) \\ &\quad \times [\mathcal{E}(r_s, \ell_2) + i\mathcal{B}(r_s, \ell_2)] e^{-2i\varphi_{\ell_2}},\end{aligned}\tag{A.1}$$



$$\begin{aligned}
\mathcal{D}^{(2)}(\ell) &= \frac{1}{2\pi} \int d^2x \theta^{a(2)} \nabla_a \mathcal{P} e^{i\ell \cdot \mathbf{x}} \\
&= -\frac{1}{\pi} \int d^2\ell_2 [(\ell - \ell_2) \cdot \ell_2] \int_0^{r_s} dr \frac{r_s - r}{r_s r} \Phi_W^{(2)}(r, \ell - \ell_2) \\
&\quad \times [\mathcal{E}(r_s, \ell_2) + i\mathcal{B}(r_s, \ell_2)] e^{-2i\varphi_{\ell_2}} \\
&\quad + \frac{1}{\pi^2} \int d^2\ell_2 \int d^2\ell_3 [(\ell + \ell_2 - \ell_3) \cdot \ell_3] [(\ell + \ell_2 - \ell_3) \cdot \ell_2] \\
&\quad \times \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^r dr' \frac{r - r'}{r r'} \Phi_W(r, \ell + \ell_2 - \ell_3) \bar{\Phi}_W(r', \ell_2) \\
&\quad \times [\mathcal{E}(r_s, \ell_3) + i\mathcal{B}(r_s, \ell_3)] e^{-2i\varphi_{\ell_3}}, \tag{A.2}
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}^{(11)}(\ell) &= \frac{1}{2\pi} \int d^2x \frac{1}{2} \theta^{a(1)} \theta^{b(1)} \nabla_a \nabla_b \mathcal{P} e^{i\ell \cdot \mathbf{x}} \\
&= \frac{1}{2} \frac{1}{\pi^2} \int d^2\ell_2 \int d^2\ell_3 [(\ell + \ell_2 - \ell_3) \cdot \ell_3] (\ell_2 \cdot \ell_3) \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^{r_s} dr' \frac{r_s - r'}{r_s r'} \\
&\quad \times \Phi_W(r, \ell + \ell_2 - \ell_3) \bar{\Phi}_W(r', \ell_2) [\mathcal{E}(r_s, \ell_3) + i\mathcal{B}(r_s, \ell_3)] e^{-2i\varphi_{\ell_3}}, \tag{A.3}
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}^{(3)}(\ell) &= \frac{1}{2\pi} \int d^2x \theta^{a(3)} \nabla_a \mathcal{P} e^{i\ell \cdot \mathbf{x}} \\
&= -\frac{1}{\pi} \int d^2\ell_2 [(\ell - \ell_2) \cdot \ell_2] \int_0^{r_s} dr \frac{r_s - r}{r_s r} \Phi_W^{(3)}(r, \ell - \ell_2) \\
&\quad \times [\mathcal{E}(r_s, \ell_2) + i\mathcal{B}(r_s, \ell_2)] e^{-2i\varphi_{\ell_2}} \\
&\quad + \frac{1}{\pi^2} \int d^2\ell_2 \int d^2\ell_3 [(\ell + \ell_2 - \ell_3) \cdot \ell_3] [(\ell + \ell_2 - \ell_3) \cdot \ell_2] \int_0^{r_s} dr' \frac{r_s - r'}{r_s r'} \\
&\quad \times \int_0^r dr' \frac{r - r'}{r r'} \left[ \Phi_W(r, \ell + \ell_2 - \ell_3) \bar{\Phi}_W^{(2)}(r', \ell_2) \right. \\
&\quad \left. + \Phi_W^{(2)}(r, \ell + \ell_2 - \ell_3) \bar{\Phi}_W(r', \ell_2) \right] [\mathcal{E}(r_s, \ell_3) + i\mathcal{B}(r_s, \ell_3)] e^{-2i\varphi_{\ell_3}} \\
&\quad - \frac{1}{\pi^3} \int d^2\ell_2 \int d^2\ell_3 \int d^2\ell_4 \{ [(\ell - \ell_2 - \ell_3 - \ell_4) \cdot \ell_4] [(\ell - \ell_2 - \ell_3 - \ell_4) \cdot \ell_2] \\
&\quad \times (\ell_2 \cdot \ell_3) \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^r dr' \frac{r - r'}{r r'} \int_0^{r'} dr'' \frac{r' - r''}{r' r''} \\
&\quad \times \Phi_W(r, \ell - \ell_2 - \ell_3 - \ell_4) \Phi_W(r', \ell_2) \Phi_W(r'', \ell_3) [\mathcal{E}(r_s, \ell_4) + i\mathcal{B}(r_s, \ell_4)] e^{-2i\varphi_{\ell_4}} \\
&\quad + \frac{1}{2} [(\ell - \ell_2 - \ell_3 - \ell_4) \cdot \ell_4] [(\ell - \ell_2 - \ell_3 - \ell_4) \cdot \ell_2] [(\ell - \ell_2 - \ell_3 - \ell_4) \cdot \ell_3] \\
&\quad \times \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^r dr' \frac{r - r'}{r r'} \int_0^r dr'' \frac{r - r''}{r r''} \\
&\quad \times \Phi_W(r, \ell - \ell_2 - \ell_3 - \ell_4) \Phi_W(r', \ell_2) \Phi_W(r'', \ell_3) [\mathcal{E}(r_s, \ell_4) + i\mathcal{B}(r_s, \ell_4)] e^{-2i\varphi_{\ell_4}} \}, \tag{A.4}
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}^{(12)}(\ell) &= \frac{1}{2\pi} \int d^2x \theta^{a(1)} \theta^{b(2)} \nabla_a \nabla_b \mathcal{P} e^{i\ell \cdot \mathbf{x}} \\
&= \frac{1}{\pi^2} \int d^2\ell_2 \int d^2\ell_3 [(\ell + \ell_2 - \ell_3) \cdot \ell_3] (\ell_2 \cdot \ell_3) \int_0^{r_s} dr \frac{r_s - r}{r_s r} \\
&\quad \times \int_0^{r_s} dr' \frac{r_s - r'}{r_s r'} \Phi_W(r, \ell + \ell_2 - \ell_3) \bar{\Phi}_W^{(2)}(r', \ell_2) [\mathcal{E}(r_s, \ell_3) + i\mathcal{B}(r_s, \ell_3)] e^{-2i\varphi_{\ell_3}} \\
&\quad - \frac{1}{\pi^3} \int d^2\ell_2 \int d^2\ell_3 \int d^2\ell_4 [(\ell - \ell_2 - \ell_3 - \ell_4) \cdot \ell_4] (\ell_4 \cdot \ell_2) (\ell_3 \cdot \ell_2) \\
&\quad \times \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^{r_s} dr' \frac{r_s - r'}{r_s r'} \int_0^{r'} dr'' \frac{r' - r''}{r' r''} \Phi_W(r, \ell - \ell_2 - \ell_3 - \ell_4) \\
&\quad \times \Phi_W(r', \ell_2) \Phi_W(r'', \ell_3) [\mathcal{E}(r_s, \ell_4) + i\mathcal{B}(r_s, \ell_4)] e^{-2i\varphi_{\ell_4}} \tag{A.5}
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}^{(111)}(\ell) &= \frac{1}{2\pi} \int d^2x \frac{1}{6} \theta^{a(1)} \theta^{b(1)} \theta^{c(1)} \nabla_a \nabla_b \nabla_c \mathcal{P} e^{i\ell \cdot \mathbf{x}} \\
&= -\frac{1}{6} \frac{1}{\pi^3} \int d^2\ell_2 \int d^2\ell_3 \int d^2\ell_4 [(\ell - \ell_2 - \ell_3 - \ell_4) \cdot \ell_4] (\ell_2 \cdot \ell_4) (\ell_3 \cdot \ell_4) \\
&\quad \times \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^{r_s} dr' \frac{r_s - r'}{r_s r'} \int_0^{r_s} dr'' \frac{r_s - r''}{r_s r''} \Phi_W(r, \ell - \ell_2 - \ell_3 - \ell_4) \\
&\quad \times \Phi_W(r', \ell_2) \Phi_W(r'', \ell_3) [\mathcal{E}(r_s, \ell_4) + i\mathcal{B}(r_s, \ell_4)] e^{-2i\varphi_{\ell_4}}, \tag{A.6}
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}^{(22)}(\ell) &= \frac{1}{2\pi} \int d^2x \frac{1}{2} \theta^{a(2)} \theta^{b(2)} \nabla_a \nabla_b \mathcal{P} e^{i\ell \cdot \mathbf{x}} \\
&= \frac{1}{2} \frac{1}{\pi^2} \int d^2\ell_2 \int d^2\ell_3 [(\ell + \ell_2 - \ell_3) \cdot \ell_3] (\ell_2 \cdot \ell_3) \\
&\quad \times \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^{r_s} dr' \frac{r_s - r'}{r_s r'} \Phi_W^{(2)}(r, \ell + \ell_2 - \ell_3) \bar{\Phi}_W^{(2)}(r', \ell_2) \\
&\quad \times [\mathcal{E}(r_s, \ell_3) + i\mathcal{B}(r_s, \ell_3)] e^{-2i\varphi_{\ell_3}} \\
&\quad - \frac{1}{\pi^3} \int d^2\ell_2 \int d^2\ell_3 \int d^2\ell_4 [(\ell - \ell_2 - \ell_3 - \ell_4) \cdot \ell_4] (\ell_4 \cdot \ell_2) (\ell_3 \cdot \ell_2) \\
&\quad \times \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^{r_s} dr' \frac{r_s - r'}{r_s r'} \int_0^{r'} dr'' \frac{r' - r''}{r' r''} \\
&\quad \times \Phi_W^{(2)}(r, \ell - \ell_2 - \ell_3 - \ell_4) \Phi_W(r', \ell_2) \Phi_W(r'', \ell_3) [\mathcal{E}(r_s, \ell_4) + i\mathcal{B}(r_s, \ell_4)] e^{-2i\varphi_{\ell_4}} \\
&\quad - \frac{1}{2} \frac{1}{\pi^4} \int d^2\ell_2 \int d^2\ell_3 \int d^2\ell_4 \int d^2\ell_5 [(\ell - \ell_2 - \ell_3 - \ell_4 - \ell_5) \cdot \ell_5] \\
&\quad \times [(\ell - \ell_2 - \ell_3 - \ell_4 - \ell_5) \cdot \ell_2] (\ell_5 \cdot \ell_3) (\ell_3 \cdot \ell_4) \int_0^{r_s} dr \frac{r_s - r}{r_s r} \\
&\quad \times \int_0^r dr' \frac{r - r'}{r r'} \int_0^{r_s} dr'' \frac{r_s - r''}{r_s r''} \int_0^{r''} dr''' \frac{r'' - r'''}{r'' r'''} \\
&\quad \times \Phi_W(r, \ell - \ell_2 - \ell_3 - \ell_4 - \ell_5) \Phi_W(r', \ell_2) \Phi_W(r'', \ell_3) \Phi_W(r''', \ell_4) \\
&\quad \times [\mathcal{E}(r_s, \ell_5) + i\mathcal{B}(r_s, \ell_5)] e^{-2i\varphi_{\ell_5}}, \tag{A.7}
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}^{(13)}(\ell) &= \frac{1}{2\pi} \int d^2x \theta^{a(1)} \theta^{b(3)} \nabla_a \nabla_b \mathcal{P} e^{i\ell \cdot \mathbf{x}} \\
&= \frac{1}{\pi^2} \int d^2\ell_2 \int d^2\ell_3 [(\ell + \ell_2 - \ell_3) \cdot \ell_3] (\ell_2 \cdot \ell_3) \\
&\quad \times \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^{r_s} dr' \frac{r_s - r'}{r_s r'} \Phi_W(r, \ell + \ell_2 - \ell_3) \bar{\Phi}_W^{(3)}(r', \ell_3) \\
&\quad \times [\mathcal{E}(r_s, \ell_3) + i\mathcal{B}(r_s, \ell_3)] e^{-2i\varphi_{\ell_3}} \\
&\quad - \frac{1}{\pi^3} \int d^2\ell_2 \int d^2\ell_3 \int d^2\ell_4 [(\ell - \ell_2 - \ell_3 - \ell_4) \cdot \ell_4] (\ell_4 \cdot \ell_2) (\ell_3 \cdot \ell_2) \\
&\quad \times \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^{r_s} dr' \frac{r_s - r'}{r_s r'} \int_0^{r'} dr'' \frac{r' - r''}{r' r''} \\
&\quad \times \Phi_W(r, \ell - \ell_2 - \ell_3 - \ell_4) \left[ \Phi_W(r', \ell_2) \Phi_W^{(2)}(r'', \ell_3) \right. \\
&\quad \left. + \Phi_W^{(2)}(r', \ell_2) \Phi_W(r'', \ell_3) \right] [\mathcal{E}(r_s, \ell_4) + i\mathcal{B}(r_s, \ell_4)] e^{-2i\varphi_{\ell_4}} \\
&\quad - \frac{1}{\pi^4} \int d^2\ell_2 \int d^2\ell_3 \int d^2\ell_4 \int d^2\ell_5 \{ [(\ell - \ell_2 - \ell_3 - \ell_4 - \ell_5) \cdot \ell_5] (\ell_2 \cdot \ell_5) \\
&\quad \times (\ell_2 \cdot \ell_3) (\ell_3 \cdot \ell_4) \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^{r_s} dr' \frac{r_s - r'}{r_s r'} \int_0^{r'} dr'' \frac{r' - r''}{r' r''} \\
&\quad \times \int_0^{r''} dr''' \frac{r'' - r'''}{r'' r'''} \Phi_W(r, \ell - \ell_2 - \ell_3 - \ell_4 - \ell_5) \Phi_W(r', \ell_2) \\
&\quad \times \Phi_W(r'', \ell_3) \Phi_W(r''', \ell_4) [\mathcal{E}(r_s, \ell_5) + i\mathcal{B}(r_s, \ell_5)] e^{-2i\varphi_{\ell_5}} \\
&\quad + \frac{1}{2} [(\ell - \ell_2 - \ell_3 - \ell_4 - \ell_5) \cdot \ell_5] (\ell_2 \cdot \ell_5) (\ell_2 \cdot \ell_3) (\ell_2 \cdot \ell_4) \int_0^{r_s} dr \frac{r_s - r}{r_s r} \\
&\quad \times \int_0^{r_s} dr' \frac{r_s - r'}{r_s r'} \int_0^{r'} dr'' \frac{r' - r''}{r' r''} \int_0^{r''} dr''' \frac{r'' - r'''}{r'' r'''} \\
&\quad \times \Phi_W(r, \ell - \ell_2 - \ell_3 - \ell_4 - \ell_5) \Phi_W(r', \ell_2) \Phi_W(r'', \ell_3) \Phi_W(r''', \ell_4) \\
&\quad \times [\mathcal{E}(r_s, \ell_5) + i\mathcal{B}(r_s, \ell_5)] e^{-2i\varphi_{\ell_5}} \} , \tag{A.8}
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}^{(1111)}(\ell) &= \frac{1}{2\pi} \int d^2x \frac{1}{24} \theta^{a(1)} \theta^{b(1)} \theta^{c(1)} \theta^{d(1)} \nabla_a \nabla_b \nabla_c \nabla_d \mathcal{P} e^{i\ell \cdot \mathbf{x}} \\
&= -\frac{1}{24} \frac{1}{\pi^4} \int d^2\ell_2 \int d^2\ell_3 \int d^2\ell_4 \int d^2\ell_5 [(\ell - \ell_2 - \ell_3 - \ell_4 - \ell_5) \cdot \ell_5] \\
&\quad \times (\ell_2 \cdot \ell_5) (\ell_3 \cdot \ell_5) (\ell_4 \cdot \ell_5) \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^{r_s} dr' \frac{r_s - r'}{r_s r'} \int_0^{r_s} dr'' \frac{r_s - r''}{r_s r''} \\
&\quad \times \int_0^{r_s} dr''' \frac{r_s - r'''}{r_s r'''} \Phi_W(r, \ell - \ell_2 - \ell_3 - \ell_4 - \ell_5) \Phi_W(r', \ell_2) \Phi_W(r'', \ell_3) \\
&\quad \times \Phi_W(r''', \ell_4) [\mathcal{E}(r_s, \ell_5) + i\mathcal{B}(r_s, \ell_5)] e^{-2i\varphi_{\ell_5}} . \tag{A.9}
\end{aligned}$$

We do not write the terms  $\mathcal{D}^{(4)}$  and  $\mathcal{D}^{(112)}$  because the associated contributions to the angular power spectra of lensed polarization tensor vanish as a consequence of statistical isotropy (see Sect. 3).

## B Lensed angular power spectra for polarization

Following Sect. 3 and [18] we now present the evaluation of the next-to-leading order corrections to E- and B-mode polarization spectra. More details are given in Ref. [18], where we compute, however, only the temperature anisotropy spectrum. We therefore repeat the procedure here for the polarization spectra and for the temperature polarization cross-correlation for completeness.

### B.1 Results $\tilde{C}_\ell^{\mathcal{EM}}$

Let's begin evaluating the lensed cross-correlation,  $\tilde{C}_\ell^{\mathcal{EM}}$ . Up to next to next-to-leading order, we have

$$\begin{aligned} -e^{2i\varphi_\ell} \langle \tilde{\mathcal{P}}(\ell) \tilde{\mathcal{M}}(\ell') \rangle &= \delta(\ell - \ell') \tilde{C}_\ell^{\mathcal{EM}} \\ &= \delta(\ell - \ell') C_\ell^{\mathcal{EM}} - e^{2i\varphi_\ell} \langle \mathcal{D}(\ell) \bar{\mathcal{A}}(\ell') \rangle, \end{aligned} \quad (\text{B.1})$$

where  $\mathcal{A}(\ell)$  is given in Eq. (3.19) and we introduce

$$\mathcal{D}(\ell) = \mathcal{D}^{(0)}(\ell) + \sum_{i=1}^4 \mathcal{D}^{(i)}(\ell) + \sum_{\substack{i+j \leq 4 \\ 1 \leq i \leq j}} \mathcal{D}^{(ij)}(\ell) + \sum_{\substack{i+j+k \leq 4 \\ 1 \leq i \leq j \leq k}} \mathcal{D}^{(ijk)}(\ell) + \mathcal{D}^{(1111)}(\ell), \quad (\text{B.2})$$

the 2d Fourier transforms of  $\mathcal{D}(x^a)$  defined in Eq. (2.4). We now introduce the expectation values  $\hat{F}_\ell^{(i\dots)}$  and  $\hat{F}_\ell^{(i\dots, j\dots)}$  by

$$\begin{aligned} \delta(\ell - \ell') \hat{F}_\ell^{(ij\dots, ij\dots)} &= \langle \mathcal{D}^{(ij\dots)}(\ell) \bar{\mathcal{A}}^{(ij\dots)}(\ell') \rangle, \\ \delta(\ell - \ell') \hat{F}_\ell^{(ij\dots, i'j'\dots)} &= \langle \mathcal{D}^{(ij\dots)}(\ell) \bar{\mathcal{A}}^{(i'j'\dots)}(\ell') \rangle + \langle \mathcal{D}^{(i'j'\dots)}(\ell) \bar{\mathcal{A}}^{(ij\dots)}(\ell') \rangle, \end{aligned} \quad (\text{B.3})$$

where the last definition applies when the coefficients  $(ij\dots)$  and  $(i'j'\dots)$  are not identical. The delta Dirac function  $\delta(\ell - \ell')$  is a consequence of statistical isotropy. By omitting terms of higher than fourth order in the Weyl potential and terms that vanish as a consequence of Wick's theorem (odd number of Weyl potentials), we obtain

$$\begin{aligned} \tilde{C}_\ell^{\mathcal{EM}} &= C_\ell^{\mathcal{EM}} + F_\ell^{(0,2)} + F_\ell^{(0,11)} + F_\ell^{(1,1)} + F_\ell^{(0,4)} + F_\ell^{(0,13)} + F_\ell^{(0,22)} + F_\ell^{(0,112)} + F_\ell^{(0,1111)} \\ &\quad + F_\ell^{(1,3)} + F_\ell^{(2,2)} + F_\ell^{(1,12)} + F_\ell^{(1,111)} + F_\ell^{(2,11)} + F_\ell^{(11,11)}, \end{aligned} \quad (\text{B.4})$$

where  $F_\ell^{(i\dots, j\dots)} = -e^{2i\varphi_\ell} \hat{F}_\ell^{(i\dots, j\dots)}$ .

As for the case of the  $\mathcal{D}^{(i\dots)}$  terms, the  $\hat{F}_\ell^{(i\dots, j\dots)}$  terms can be easily evaluated from the  $C_\ell^{(i\dots, j\dots)}$ . In fact, using Eq. (3.13) and the results for the  $\mathcal{D}^{(i\dots)}$  and  $\mathcal{A}^{(i\dots)}$  terms (see Sect. 5, Appendix A and [18]), one finds that the  $\hat{F}_\ell^{(i\dots, j\dots)}$  are given by the  $C_\ell^{(i\dots, j\dots)}$  simply by substituting

$$C_\ell^{\mathcal{M}}(z_s) \rightarrow -C_\ell^{\mathcal{EM}}(z_s) e^{-2i\varphi_\ell}. \quad (\text{B.5})$$

The substitution is performed for any  $C_\ell^{\mathcal{M}}(z_s)$  inside and outside the integrals.

## B.2 Results $\tilde{C}_\ell^\mathcal{E} + \tilde{C}_\ell^\mathcal{B}$

Let us also evaluate  $\tilde{C}_\ell^\mathcal{E} + \tilde{C}_\ell^\mathcal{B}$ . Proceeding as in the previous subsection we have

$$\begin{aligned}\langle \tilde{\mathcal{P}}(\ell) \tilde{\mathcal{P}}(\ell') \rangle &= \delta(\ell - \ell') \left[ \tilde{C}_\ell^\mathcal{E} + \tilde{C}_\ell^\mathcal{B} \right] \\ &= \delta(\ell - \ell') \left[ C_\ell^\mathcal{E} + C_\ell^\mathcal{B} \right] + \langle \mathcal{D}(\ell) \bar{\mathcal{D}}(\ell') \rangle.\end{aligned}\quad (\text{B.6})$$

We now introduce  $M_\ell^{(i\dots)}$  and  $M_\ell^{(i\dots, j\dots)}$  given by

$$\begin{aligned}\delta(\ell - \ell') M_\ell^{(ij\dots, ij\dots)} &= \langle \mathcal{D}^{(ij\dots)}(\ell) \bar{\mathcal{D}}^{(ij\dots)}(\ell') \rangle, \\ \delta(\ell - \ell') M_\ell^{(ij\dots, i'j'\dots)} &= \langle \mathcal{D}^{(ij\dots)}(\ell) \bar{\mathcal{D}}^{(i'j'\dots)}(\ell') \rangle + \langle \mathcal{D}^{(i'j'\dots)}(\ell) \bar{\mathcal{D}}^{(ij\dots)}(\ell') \rangle,\end{aligned}\quad (\text{B.7})$$

where again the last definition applies when the coefficients  $(ij\dots)$  and  $(i'j'\dots)$  are not identical. The delta Dirac function  $\delta(\ell - \ell')$  is a consequence of statistical isotropy. As before, by omitting terms of higher than fourth order in the Weyl potential and terms that vanish as a consequence of Wick's theorem, we obtain

$$\begin{aligned}\left[ \tilde{C}_\ell^\mathcal{E} + \tilde{C}_\ell^\mathcal{B} \right] &= \left[ C_\ell^\mathcal{E} + C_\ell^\mathcal{B} \right] + M_\ell^{(0,11)} + M_\ell^{(1,1)} + M_\ell^{(0,2)} + M_\ell^{(0,13)} + M_\ell^{(0,22)} + M_\ell^{(0,112)} \\ &\quad + M_\ell^{(0,1111)} + M_\ell^{(1,3)} + M_\ell^{(2,2)} + M_\ell^{(1,12)} + M_\ell^{(1,111)} + M_\ell^{(2,11)} + M_\ell^{(11,11)}.\end{aligned}\quad (\text{B.8})$$

As for the case of the  $F_\ell^{(i\dots, j\dots)}$  terms, also in this case we can obtain the  $M_\ell^{(i\dots, j\dots)}$  terms starting from the results for the  $C_\ell^{(i\dots, j\dots)}$ . These will be obtained by the  $C_\ell^{(i\dots, j\dots)}$  via the substitution

$$C_\ell^\mathcal{M}(z_s) \rightarrow C_\ell^\mathcal{E}(z_s) + C_\ell^\mathcal{B}(z_s), \quad (\text{B.9})$$

performed for any  $C_\ell^\mathcal{M}(z_s)$  inside and outside the integrals.

## B.3 Results $\tilde{C}_\ell^\mathcal{E} - \tilde{C}_\ell^\mathcal{B}$

Let us finally move to the evaluation of  $\tilde{C}_\ell^\mathcal{E} - \tilde{C}_\ell^\mathcal{B}$ . Proceeding as in the previous subsections we have

$$\begin{aligned}\langle \tilde{\mathcal{P}}(\ell) \tilde{\mathcal{P}}(\ell') \rangle &= \delta(\ell + \ell') \left[ \tilde{C}_\ell^\mathcal{E} - \tilde{C}_\ell^\mathcal{B} \right] e^{-4i\varphi_\ell} \\ &= \delta(\ell + \ell') \left[ C_\ell^\mathcal{E} - C_\ell^\mathcal{B} \right] e^{-4i\varphi_\ell} + \langle \mathcal{D}(\ell) \mathcal{D}(\ell') \rangle.\end{aligned}\quad (\text{B.10})$$

We now introduce  $\hat{N}_\ell^{(i\dots, j\dots)}$  defined as follows

$$\begin{aligned}\delta(\ell + \ell') \hat{N}_\ell^{(ij\dots, ij\dots)} &= \langle \mathcal{D}^{(ij\dots)}(\ell) \mathcal{D}^{(ij\dots)}(\ell') \rangle, \\ \delta(\ell + \ell') \hat{N}_\ell^{(ij\dots, i'j'\dots)} &= \langle \mathcal{D}^{(ij\dots)}(\ell) \mathcal{D}^{(i'j'\dots)}(\ell') \rangle + \langle \mathcal{D}^{(i'j'\dots)}(\ell) \mathcal{D}^{(ij\dots)}(\ell') \rangle,\end{aligned}\quad (\text{B.11})$$

where the last definition applies when the coefficients  $(ij\dots)$  and  $(i'j'\dots)$  are different. The  $\delta(\ell + \ell')$  is a consequence of statistical isotropy and of the fact that in general  $A(\ell) = \bar{A}(-\ell)$ .

As before, by omitting terms of higher than fourth order in the Weyl potential and terms that vanish as a consequence of Wick's theorem, we obtain

$$\begin{aligned} [\tilde{C}_\ell^{\mathcal{E}} - \tilde{C}_\ell^{\mathcal{B}}] = [C_\ell^{\mathcal{E}} - C_\ell^{\mathcal{B}}] + N_\ell^{(0,2)} + N_\ell^{(0,11)} + N_\ell^{(1,1)} + N_\ell^{(0,4)} + N_\ell^{(0,13)} + N_\ell^{(0,22)} + N_\ell^{(0,112)} \\ + N_\ell^{(0,1111)} + N_\ell^{(1,3)} + N_\ell^{(2,2)} + N_\ell^{(1,12)} + N_\ell^{(1,111)} + N_\ell^{(2,11)} + N_\ell^{(11,11)}, \end{aligned} \quad (\text{B.12})$$

where  $N_\ell^{(i,\dots,j,\dots)} = e^{4i\varphi_\ell} \hat{N}_\ell^{(i,\dots,j,\dots)}$

Like for the other terms, we can obtain the  $\hat{N}_\ell^{(i,\dots,j,\dots)}$  terms starting from the results for the  $C_\ell^{(i,\dots,j,\dots)}$  by substituting

$$C_\ell^{\mathcal{M}}(z_s) \rightarrow [C_\ell^{\mathcal{E}}(z_s) - C_\ell^{\mathcal{B}}(z_s)] e^{-4\varphi_\ell}, \quad (\text{B.13})$$

for any  $C_\ell^{\mathcal{M}}(z_s)$  inside and outside the integrals.

Using these results we obtain the corrections to the different polarization power spectra. The general rules to follow are specified in Eqs. (3.15)-(3.17).

## C Rotation angle using the Sachs formalism

In this Appendix we determine the rotation angle via the Sachs basis and show that the result obtained is equivalent to the rotation angle through the amplification matrix.

For this purpose, we work in the so called geodesic light-cone (GLC) coordinates [42]. They consist of a timelike coordinate  $\tau$  (which can always be identified with the proper time in the synchronous gauge [43]), a null coordinate  $w$ , and two angular coordinates  $\tilde{\theta}^a$  ( $a = 1, 2$ ). The GLC line-element depends on six arbitrary functions ( $\Upsilon, U^a, \gamma_{ab} = \gamma_{ba}$ ), and takes the form

$$ds^2 = \Upsilon^2 dw^2 - 2\Upsilon dw d\tau + \gamma_{ab}(d\tilde{\theta}^a - U^a dw)(d\tilde{\theta}^b - U^b dw) \quad , \quad a, b = 1, 2 \quad (\text{C.1})$$

where  $\gamma_{ab}$  and its inverse  $\gamma^{ab}$  lower and raise two-dimensional indices. In the GLC coordinates the past light-cone of a given observer is defined by  $w = w_o = \text{constant}$ , and null geodesics stay at fixed values of the angular coordinates  $\tilde{\theta}^a = \tilde{\theta}_o^a = \text{constant}$  (with  $\tilde{\theta}_o^a$  specifying the direction of observation). In this system of coordinates the so-called screen, normal to incoming photon geodesics and the observers worldline, is simply given by the 2-dimensional subspace spanned by the angles  $\tilde{\theta}^a$ . The Sachs basis is determined up to a global rotation by the equations [44]

$$\gamma_{ab} s_A^a s_B^b = \delta_{AB} \quad , \quad \nabla_\tau s_A^a = 0. \quad (\text{C.2})$$

To zeroth order

$$\left(\gamma_{ab}^{(0)}\right) = a^2(\tau) r^2(\tau, w) \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \tilde{\theta}^1 \end{pmatrix}, \quad (\text{C.3})$$

$$(s_A^a)^{(0)} = [a(\tau) r(\tau, w)]^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \sin^{-1} \tilde{\theta}^1 \end{pmatrix}. \quad (\text{C.4})$$

Due to the factorization of the time dependence, we have that  $\partial_\tau (s_A^a)^{(0)} \propto (s_A^a)^{(0)}$  and  $\partial_\tau \gamma_{ab}^{(0)} \propto \gamma_{ab}^{(0)}$ . At first order,  $\gamma_{ab} = \gamma_{ab}^{(0)} + \gamma_{ab}^{(1)}$  and  $s_A^a = (s_A^a)^{(0)} + (s_A^a)^{(1)}$ , the normalisation condition yields

$$(s_A^c)^{(1)} + \gamma_{ab}^{(0)} (s_A^a)^{(0)} (s_B^b)^{(1)} (s_B^c)^{(0)} = -\gamma_{(0)}^{cb} \gamma_{ba}^{(1)} (s_A^a)^{(0)}, \quad (\text{C.5})$$

which is satisfied by

$$(s_A^c)^{(1)} = -\frac{1}{2} \gamma_{(0)}^{cb} \gamma_{ba}^{(1)} (s_A^a)^{(0)} \quad (\text{C.6})$$

or

$$s_{aA}^{(1)} = \frac{1}{2} \gamma_{ab}^{(1)} (s_A^b)^{(0)}. \quad (\text{C.7})$$

The second condition of Eqs. (C.2) can be rewritten as  $\epsilon^{AB} \partial_\tau s_A^a s_{aB} = 0$ , where  $\epsilon^{AB}$  is the Levi-Civita symbol in flat space. This equation is also satisfied by Eq. (C.6). Note that an arbitrary orthonormal basis of the screen allows a residual freedom of rotation given by  $\mathcal{R} \in SO(2)$ . Indeed, if  $s_A^a$  is a solution of  $\gamma_{ab} s_A^a s_B^b = \delta_{AB}$ ,  $\tilde{s}_A^a = \mathcal{R}_A^B s_B^a$  is solution too, where

$$\mathcal{R}_A^B = \begin{pmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{pmatrix}, \quad (\text{C.8})$$

with an arbitrary rotation angle  $\beta$ . In this way, an appropriate choice of a time-dependent rotation angle  $\beta$  allows us to satisfy both conditions in Eqs. (C.2). Starting from a generic orthonormal zweibein, we choose the rotation  $\mathcal{R}$  such that the rotated zweibein is parallel transported along lightlike geodesics. To achieve this the rotation angle  $\beta$  has to satisfy the relation

$$\partial_\tau \beta = \frac{1}{2} \epsilon^{AB} \partial_\tau s_A^a s_{aB}, \quad (\text{C.9})$$

see also Appendix A of [44]. In [45], an exact expression of  $\beta$  was obtained in this context (see Eqs. (A.3)-(A.4)).

For our purpose, we expand  $\beta$  in Eq. (C.8) up to fourth order, since in principle we require the rotation of the Sachs basis up to fourth order to compute all the contributions to the next-to-leading order of the polarization spectra, i.e.  $\beta = \beta^{(0)} + \beta^{(1)} + \beta^{(2)} + \beta^{(3)} + \beta^{(4)}$ . Nevertheless, we shall show that  $\beta^{(0)}$  and  $\beta^{(1)}$  do not induce a local rotation of the basis, so that they can be set to zero. Therefore, we can write the rotation matrix up to fourth order as

$$\mathcal{R}_A^B = \left[ 1 - \frac{(\beta^{(2)})^2}{2} \right] \delta_A^B + \left( \beta^{(2)} + \beta^{(3)} + \beta^{(4)} \right) \epsilon_A^B. \quad (\text{C.10})$$

Hence the parallel transported Sachs basis is

$$\begin{aligned}
\tilde{s}_A^a &= \mathcal{R}_A{}^B s_B^a = \left\{ \left[ 1 - \frac{(\beta^{(2)})^2}{2} \right] \delta_A^B + \left( \beta^{(2)} + \beta^{(3)} + \beta^{(4)} \right) \epsilon_A{}^B \right\} \\
&\quad \times \left[ (s_B^a)^{(0)} + (s_B^a)^{(1)} + (s_B^a)^{(2)} + (s_B^a)^{(3)} + (s_B^a)^{(4)} \right] \\
&= s_A^a - \frac{(\beta^{(2)})^2}{2} (s_A^a)^{(0)} + \beta^{(2)} \epsilon_A{}^B \left[ (s_B^a)^{(0)} + (s_B^a)^{(1)} + (s_B^a)^{(2)} \right] \\
&\quad + \beta^{(3)} \epsilon_A{}^B \left[ (s_B^a)^{(0)} + (s_B^a)^{(1)} \right] + \beta^{(4)} \epsilon_A{}^B (s_B^a)^{(0)}, \tag{C.11}
\end{aligned}$$

where  $(s_B^a)$  is an arbitrary normalized zweibein on the screen. In the main text we note that  $\beta^{(3)}$  and  $\beta^{(4)}$  do not contribute at next to leading order for statistical isotropy, we can thus just focus on determining  $\beta^{(2)}$ .

Before that, we prove that the solution (C.6) combined with Eq. (C.9) implies  $\beta^{(1)} = \text{constant}$ . Of course  $\beta^{(0)}$  is constant since our background is isotropic. Indeed, Eq. (C.9) for the background yields

$$\partial_\tau \beta^{(0)} = \frac{1}{2} \epsilon^{AB} \partial_\tau (s_A^a)^{(0)} (s_{aB})^{(0)} \propto \epsilon^{AB} (s_A^a)^{(0)} (s_{aB})^{(0)} = \epsilon^{AB} \delta_{AB} = 0, \tag{C.12}$$

because  $\epsilon^{AB}$  is antisymmetric whereas  $\delta_{AB}$  is symmetric. With a global rotation we can choose  $\beta^{(0)} = 0$ . In the same way, we can show that  $\partial_\tau \beta^{(1)}$  vanishes. We have that

$$\begin{aligned}
\partial_\tau \beta^{(1)} &= -\frac{1}{4} \epsilon^{AB} \partial_\tau \gamma_{(0)}^{ab} \gamma_{bc}^{(1)} (s_A^c)^{(0)} (s_{aB})^{(0)} - \frac{1}{4} \epsilon^{AB} \gamma_{(0)}^{ab} \partial_\tau \gamma_{bc}^{(1)} (s_A^c)^{(0)} (s_{aB})^{(0)} \\
&\quad - \frac{1}{4} \epsilon^{AB} \gamma_{(0)}^{ab} \gamma_{bc}^{(1)} \partial_\tau (s_A^c)^{(0)} (s_{aB})^{(0)} + \frac{1}{4} \epsilon^{AB} \partial_\tau (s_A^a)^{(0)} \gamma_{ab}^{(1)} (s_B^b)^{(0)}. \tag{C.13}
\end{aligned}$$

Considering that the last two terms cancel and using  $\epsilon^{AB} (s_A^a)^{(0)} (s_B^b)^{(0)} \propto \epsilon^{ab}$  [45], we obtain

$$\partial_\tau \beta^{(1)} = -F \frac{1}{4} \epsilon^{cd} \partial_\tau \gamma_{(0)}^{ab} \gamma_{bc}^{(1)} \gamma_{da}^{(0)} - G \frac{1}{4} \epsilon^{cb} \partial_\tau \gamma_{bc}^{(1)}, \tag{C.14}$$

which vanish separately for arbitrary functions  $F$  and  $G$  as in both cases the epsilon tensor is contracted with a symmetric expression.

Let us now determine the second-order contribution to the Sachs basis. The orthogonality condition

$$(s_A^c)^{(2)} + \gamma_{ab}^{(0)} (s_A^a)^{(0)} (s_B^b)^{(2)} (s_B^c)^{(0)} = \frac{3}{4} (s_A^a)^{(0)} \gamma_{ab}^{(1)} \gamma_{(0)}^{bd} \gamma_{de}^{(1)} \gamma_{(0)}^{ec} - (s_A^a)^{(0)} \gamma_{ab}^{(2)} \gamma_{(0)}^{bc} \tag{C.15}$$

is satisfied by the solution

$$(s_A^a)^{(2)} = \left( -\frac{1}{2} \gamma_{(0)}^{ab} \gamma_{be}^{(2)} + \frac{3}{8} \gamma_{(0)}^{ab} \gamma_{bc}^{(1)} \gamma_{(0)}^{cd} \gamma_{de}^{(1)} \right) (s_A^e)^{(0)} \tag{C.16}$$

or

$$(s_a^A)^{(2)} = \left( \frac{1}{2} \gamma_{ad}^{(2)} - \frac{1}{8} \gamma_{ab}^{(1)} \gamma_0^{bc} \gamma_{cd}^{(1)} \right) (s_A^d)^{(0)}. \tag{C.17}$$



This orthonormal basis of the screen still has to be rotated by some angle  $\beta^{(2)}$  in order to be parallel transported. We now compute this rotation angle using Eq. (C.9). At second order it yields

$$\partial_\tau \beta^{(2)} = \frac{1}{2} \epsilon^{AB} \left[ \partial_\tau (s_A^a)^{(2)} s_{aB}^{(0)} + \partial_\tau (s_A^a)^{(0)} s_{aB}^{(2)} + \partial_\tau (s_A^a)^{(1)} s_{aB}^{(1)} \right]. \quad (\text{C.18})$$

It is easy to verify that first and second term on the rhs of Eq. (C.18) cancel just as for the first order rotation angle. We focus on the remaining term:

$$\begin{aligned} \epsilon^{AB} \partial_\tau (s_A^a)^{(1)} s_{aB}^{(1)} &= -\frac{1}{4} \epsilon^{AB} \partial_\tau \gamma_{(0)}^{ab} \gamma_{bc}^{(1)} (s_A^c)^{(0)} \gamma_{ad}^{(1)} (s_B^d)^{(0)} - \frac{1}{4} \epsilon^{AB} \gamma_{(0)}^{ab} \partial_\tau \gamma_{bc}^{(1)} (s_A^c)^{(0)} \gamma_{ad}^{(1)} (s_B^d)^{(0)} \\ &\quad - \frac{1}{4} \epsilon^{AB} \gamma_{(0)}^{ab} \gamma_{bc}^{(1)} \partial_\tau (s_A^c)^{(0)} \gamma_{ad}^{(1)} (s_B^d)^{(0)}. \end{aligned} \quad (\text{C.19})$$

Using the identities  $\epsilon^{AB} (s_A^a)^{(0)} (s_B^b)^{(0)} = \gamma_{(0)}^{-1/2} \epsilon^{ab}$ , with  $\det \gamma_{ab}^{(0)} \equiv \gamma_{(0)}$  and  $\partial_\tau (s_A^a)^{(0)} = -\frac{1}{4} \frac{\partial_\tau \gamma_{(0)}}{\gamma_{(0)}} (s_A^a)^{(0)}$ , as well as the antisymmetry of  $\epsilon^{cd}$ , Eq. (C.19) simplifies to

$$\epsilon^{AB} \partial_\tau (s_A^a)^{(1)} s_{aB}^{(1)} = -\frac{1}{4} \gamma_{(0)}^{-1/2} \epsilon^{ab} \partial_\tau \gamma_{bc}^{(1)} \epsilon^{cd} \gamma_{da}^{(1)}. \quad (\text{C.20})$$

Hence

$$\partial_\tau \beta^{(2)} = -\frac{1}{8} \gamma_{(0)}^{-1/2} \epsilon^{ab} \partial_\tau \gamma_{bc}^{(1)} \epsilon^{cd} \gamma_{da}^{(1)}. \quad (\text{C.21})$$

The first order perturbations of the angular part of the metric,  $\gamma_{ab}^{(1)}$  can be expressed in terms of the first order deflection angle, see [26]

$$\gamma_{ab}^{(1)} = \gamma_{ac}^{(0)} \partial_b \theta^{c(1)} + \gamma_{cb}^{(0)} \partial_a \theta^{c(1)}. \quad (\text{C.22})$$

Using also  $\partial_\tau \gamma_{ab}^{(0)} = \frac{1}{2} \frac{\partial_\tau \gamma_{(0)}}{\gamma_{(0)}} \gamma_{ab}^{(0)}$ , we obtain the second order rotation in terms of first order deflection angles,

$$\begin{aligned} \partial_\tau \beta^{(2)} &= -\frac{1}{8} \gamma_{(0)}^{-1/2} \partial_c \partial_\tau \theta^{a(1)} \epsilon^{cd} \gamma_{da}^{(1)} - \frac{1}{8} \gamma_{(0)}^{-1/2} \gamma_{(0)}^{ab} \gamma_{ce}^{(0)} \partial_b \partial_\tau \theta^{e(1)} \epsilon^{cd} \gamma_{da}^{(1)} \\ &\quad - \frac{1}{16} \frac{\partial_\tau \gamma_{(0)}}{\gamma_{(0)}^{3/2}} \partial_c \theta^{a(1)} \epsilon^{cd} \gamma_{da}^{(1)} - \frac{1}{16} \frac{\partial_\tau \gamma_{(0)}}{\gamma_{(0)}^{3/2}} \gamma_{(0)}^{ab} \gamma_{ce}^{(0)} \partial_b \theta^{e(1)} \epsilon^{cd} \gamma_{da}^{(1)}. \end{aligned} \quad (\text{C.23})$$

We finally express the rotation angle in term of the Weyl potential. Using the expression for the deflection angle given in the main text, Eq. (2.9), we obtain

$$\begin{aligned} \partial_\tau \beta^{(2)} &= a^2 \gamma_{(0)}^{-1/2} \epsilon^{ab} \gamma_{bc}^{(0)} \gamma_{(0)}^{de} \partial_d \int_\eta^{\eta_o} d\eta_1 a^2(\eta_1) \gamma_{(0)}^{cf}(\eta_1) \int_{\eta_1}^{\eta_o} d\eta_2 \partial_f \Phi_W(\eta_2) \partial_e \partial_a \int_\eta^{\eta_o} d\eta_3 \Phi_W(\eta_3) \\ &\quad + a^2 \gamma_{(0)}^{-1/2} \epsilon^{ab} \partial_b \int_\eta^{\eta_o} d\eta_1 a^2(\eta_1) \gamma_{(0)}^{cd}(\eta_1) \int_{\eta_1}^{\eta_o} d\eta_2 \partial_d \Phi_W(\eta_2) \partial_c \partial_a \int_\eta^{\eta_o} d\eta_3 \Phi_W(\eta_3). \end{aligned} \quad (\text{C.24})$$

Note that here  $\Phi_W(\eta_i) \equiv \Phi_W(\eta_i, \mathbf{n}(\eta_o - \eta_i))$  where  $\eta_o$  is present time and  $\mathbf{n}$  is the directions of the geodesic given by  $\tilde{\theta}^a$ . This expression can be further simplified using  $r_i \equiv \eta_o - \eta_i$  and  $\gamma_{(0)}^{ab} = [a(\tau) r(\tau, w)]^{-2} \hat{\gamma}_{(0)}^{ab} = [a(\eta) r]^{-2} \hat{\gamma}_{(0)}^{ab}$ . We then find

$$\partial_\eta \beta^{(2)} = 2 \frac{\hat{\gamma}_{(0)}^{-1/2}}{r^2} \epsilon^{ab} \hat{\gamma}_{(0)}^{cd} \int_0^r \frac{dr_1}{r_1^2} \int_0^{r_1} dr_2 \partial_b \partial_d \Phi_W(r_2) \int_0^r dr_3 \partial_a \partial_c \Phi_W(r_3). \quad (\text{C.25})$$

Here we have used that to lowest order  $\partial_\tau = a^{-1}\partial_\eta$  and  $\Phi_W(r_i) = \Phi_W(\eta_o - r_i, \mathbf{n}r_i)$ . This result can be integrated to yield (we use  $\int_{\eta_s}^{\eta_o} d\eta = \int_0^{r_s} dr$  and adopt the boundary condition  $\beta(\eta_o) = 0$ )

$$\beta^{(2)}(r_s) = 2\epsilon^{ab} \int_0^{r_s} \frac{dr}{r^2} \int_0^r \frac{dr_1}{r_1^2} \int_0^{r_1} dr_2 \nabla_b \nabla_c \Phi_W(r_2) \int_0^r dr_3 \nabla_a \nabla^c \Phi_W(r_3). \quad (\text{C.26})$$

We now show that  $\beta^{(2)}$  agrees with the rotation angle in the deflection matrix. This is no surprise as both express the answer to the same question: how is a basis of the screen rotated when it is parallel transported along a light ray?

The angle in the deflection matrix, as defined in Eq. (2.9) of [18] can be obtained from the second order amplification matrix. In order to evaluate it we use the expression for  $\Psi_{ab}^{(2)}$  given in Eq. (2.15) of Ref. [18]

$$\begin{aligned} \omega^{(2)} &= -\frac{1}{2} \hat{\gamma}_{(0)}^{-1/2} \epsilon^{ab} \Psi_{ab}^{(2)} \\ &= 2\epsilon^{ab} \int_0^{r_s} dr \frac{r_s - r}{r_s r} \left[ \nabla_a \nabla^c \Phi_W(r) \int_0^r dr_1 \frac{r - r_1}{r r_1} \nabla_b \nabla_c \Phi_W(r_1) \right] \\ &= 2\epsilon^{ab} \int_0^{r_s} \frac{dr}{r^2} \int_0^r dr_1 \left[ \nabla_a \nabla^c \Phi_W(r_1) \int_0^{r_1} \frac{dr_2}{r_2^2} \int_0^{r_2} dr_3 \nabla_b \nabla_c \Phi_W(r_3) \right], \end{aligned} \quad (\text{C.27})$$

where we have used the relation

$$\int_0^{r_s} dr \frac{r_s - r}{r_s r} f(r) = \int_0^{r_s} \frac{dr}{r^2} \int_0^r dr_1 f(r_1) - \lim_{r \rightarrow 0} \left[ \frac{r_s - r}{r_s r} \int_0^r dr_1 f(r_1) \right], \quad (\text{C.28})$$

for both inner and outer integrals. Third line of Eq. (C.27) can be further manipulated as follows

$$\begin{aligned} \omega^{(2)} &= 2\epsilon^{ab} \int_0^{r_s} \frac{dr}{r^2} \int_0^r dr_1 \left[ \frac{d}{dr_1} \left( \int_0^{r_1} dr_4 \nabla_a \nabla^c \Phi_W(r_4) \right) \int_0^{r_1} \frac{dr_2}{r_2^2} \int_0^{r_2} dr_3 \nabla_b \nabla_c \Phi_W(r_3) \right] \\ &= 2\epsilon^{ab} \int_0^{r_s} \frac{dr}{r^2} \int_0^r dr_1 \nabla_a \nabla^c \Phi_W(r_1) \int_0^r \frac{dr_2}{r_2^2} \int_0^{r_2} dr_3 \nabla_b \nabla_c \Phi_W(r_3) \\ &\quad - 2\epsilon^{ab} \int_0^{r_s} \frac{dr}{r^2} \int_0^r dr_1 \left[ \int_0^{r_1} dr_4 \nabla_a \nabla^c \Phi_W(r_4) \frac{d}{dr_1} \left( \int_0^{r_1} \frac{dr_2}{r_2^2} \int_0^{r_2} dr_3 \nabla_b \nabla_c \Phi_W(r_3) \right) \right] \\ &= 2\epsilon^{ab} \int_0^{r_s} \frac{dr}{r^2} \int_0^r dr_1 \nabla_a \nabla^c \Phi_W(r_1) \int_0^r \frac{dr_2}{r_2^2} \int_0^{r_2} dr_3 \nabla_b \nabla_c \Phi_W(r_3) \\ &\quad - 2\epsilon^{ab} \int_0^{r_s} \frac{dr}{r^2} \int_0^r \frac{dr_1}{r_1^2} \left[ \int_0^{r_1} dr_4 \nabla_a \nabla^c \Phi_W(r_4) \int_0^{r_1} dr_3 \nabla_b \nabla_c \Phi_W(r_3) \right] \end{aligned} \quad (\text{C.29})$$

$$= \beta^{(2)} - 2\epsilon^{ab} \int_0^{r_s} \frac{dr}{r^2} \int_0^r \frac{dr_1}{r_1^2} \left[ \int_0^{r_1} dr_4 \nabla_a \nabla^c \Phi_W(r_4) \int_0^{r_1} dr_3 \nabla_b \nabla_c \Phi_W(r_3) \right]. \quad (\text{C.30})$$

To obtain (C.29), we have performed an integration by part between first and second line. The last term in Eq. (C.30) vanishes: indeed,  $\epsilon^{ab}$  is antisymmetric whereas the integral is symmetric under the exchange of  $a$  and  $b$ . This proves the equivalence of the rotation angles  $\omega^{(2)}$  and  $\beta^{(2)}$ . This equivalence should also hold at higher orders as it is the consequence of a non-perturbative equality.

We finally express this angle in  $\ell$  space. Using the flat sky approximation we expand the Weyl potential in Fourier space,

$$\Phi_W(z, \mathbf{x}) = \frac{1}{2\pi} \int d^2\ell \Phi_W(z, \ell) e^{-i\ell \cdot \mathbf{x}}. \quad (\text{C.31})$$

As in the main text, to each redshift  $z$  there corresponds a comoving distance  $r(z)$ . Inserting this expansion in Eq. (C.26) we can obtain

$$\begin{aligned} \beta^{(2)} &= \frac{2\epsilon^{ab}}{(2\pi)^2} \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int d^2\ell_1 \ell_{1a} \ell_1^c \Phi_W(z, \ell_1) e^{-i\ell_1 \cdot \mathbf{x}} \int_0^r dr_1 \frac{r - r_1}{r r_1} \\ &\quad \times \int d^2\ell_2 \ell_{2b} \ell_{2c} \Phi_W(z_1, \ell_2) e^{-i\ell_2 \cdot \mathbf{x}} \\ &= \frac{2}{(2\pi)^2} \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^r dr_1 \frac{r - r_1}{r r_1} \int d^2\ell_1 \int d^2\ell_2 \\ &\quad \times \epsilon^{ab} \ell_{1a} \ell_{2b} (\ell_1 \cdot \ell_2) \Phi_W(z, \ell_1) \Phi_W(z_1, \ell_2) e^{-i(\ell_1 + \ell_2) \cdot \mathbf{x}} \\ &= \frac{2}{(2\pi)^2} \int_0^{r_s} dr \frac{r_s - r}{r_s r} \int_0^r dr_1 \frac{r - r_1}{r r_1} \int d^2\ell_1 \int d^2\ell_2 \\ &\quad \times \mathbf{n} \cdot (\ell_2 \wedge \ell_1) (\ell_1 \cdot \ell_2) \Phi_W(z, \ell_1) \Phi_W(z_1, \ell_2) e^{-i(\ell_1 + \ell_2) \cdot \mathbf{x}}. \end{aligned} \quad (\text{C.32})$$

Here, we remember,  $\mathbf{n}$  is the direction of the light ray, orthogonal to the plane containing the  $\ell$  vectors.

## D Fisher Analysis

We briefly summarise the Fisher formalism adopted in this work to estimate the theoretical bias introduced by neglecting next-to-leading order lensing. In the ideal case of a cosmic variance limited survey, the Fisher matrix is defined by

$$F_{\alpha\beta} = \sum_{\ell} \sum_{X,Y} \frac{\partial C_{\ell}^X}{\partial q_{\alpha}} \frac{\partial C_{\ell}^Y}{\partial q_{\beta}} \text{Cov}_{\ell[X,Y]}^{-1}, \quad (\text{D.1})$$

where  $X$  and  $Y$  denote the corresponding power spectra  $(\mathcal{M}, \mathcal{E}, \mathcal{EM}, \mathcal{B})$ ,  $q_{\alpha}$  are the cosmological parameters and the covariance matrix is [46]

$$\text{Cov}_{\ell} = \frac{2}{2\ell + 1} \begin{pmatrix} (C_{\ell}^{\mathcal{M}})^2 & (C_{\ell}^{\mathcal{EM}})^2 & C_{\ell}^{\mathcal{M}} C_{\ell}^{\mathcal{EM}} & 0 \\ (C_{\ell}^{\mathcal{EM}})^2 & (C_{\ell}^{\mathcal{E}})^2 & C_{\ell}^{\mathcal{E}} C_{\ell}^{\mathcal{EM}} & 0 \\ C_{\ell}^{\mathcal{M}} C_{\ell}^{\mathcal{EM}} & C_{\ell}^{\mathcal{E}} C_{\ell}^{\mathcal{EM}} & \frac{1}{2} \left( (C_{\ell}^{\mathcal{EM}})^2 + C_{\ell}^{\mathcal{M}} C_{\ell}^{\mathcal{E}} \right) & 0 \\ 0 & 0 & 0 & (C_{\ell}^{\mathcal{B}})^2 \end{pmatrix}. \quad (\text{D.2})$$

To estimate the impact on the cosmological parameter estimation induced by neglecting a correction  $\Delta C_{\ell}$  on the leading contribution  $C_{\ell}$  we follow the formalism introduced in Refs. [47–49]. Therefore the shift of the best-fit is determined by

$$\Delta_{q_{\alpha}} = \sum_{\beta} [F^{-1}]_{\alpha\beta} B_{\beta}, \quad (\text{D.3})$$

with

$$B_\beta = \sum_\ell \sum_{X,Y} \Delta C_\ell^X \frac{\partial C_\ell^X}{\partial q_\beta} \text{Cov}_{\ell[X,Y]}^{-1}. \quad (\text{D.4})$$

Strictly speaking, a Fisher matrix analysis applies only for Gaussian distributions which is not the case of cosmological parameters in general and even less for higher order corrections. But to lowest order in the deviation from the best-fit value every statistic is Gaussian, and hence for the tiny deviations which we find a Fisher analysis is expected to be sufficient. The impact of deviation from Gaussian statistics of the lensed power spectra has been studied in [40], concluding that the errors induced on the  $(\mathcal{M}, \mathcal{E}, \mathcal{EM})$  lensed power spectra are negligible, while on B-modes the Gaussian approximation may underestimate the variance.

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